Proof Theory of Martin-Löf Type Theory

Anton Setzer

9 September 2019

We start by investigating Hilbert's program, its failure due to Gödel's Second Incompleteness Theorem and the need to replace finitary methods by extended methods. We investigate how Gentzen's cut elimination for Peano Arithmetic provides such extended finitary methods by adding transfinite induction over an ordinal notation system. We see that one can get a good insight into the well-foundedness of Gentzen's ordinal notation system. This can be extended to much bigger ordinal notation systems (the author has intuitive arguments up to the strength of $(\Pi^1_1 - \mathrm{CA})_0$). With increasing strength it becomes difficult to see the well foundedness directly (without referring to set theoretic interpretations) and therefore it is necessary to look for trustworthy theories in which one has full trust, and which can prove the well-foundedness of large ordinal notation systems and therefore the consistency of strong mathematical theories. Constructive theories are very suitable for this, and they allow to constructive classical mathematical theories. Martin Löf Type Theory is a theory where particular care has been taken into making it a trustworthy theory. We discuss as well the principle limitations of any approaches to justify the consistency of theories due to Gödel's Incompleteness Theorem.

Next we present Martin-Löf Type Theory and look for how it can be analysed proof theoretically. Obtaining lower bounds are needed in order to resolve the above problems of providing a constructive underpinning of reasonably strong classical theories. We will show how lower bounds are obtained by using Buchholz' method of distinguished sets. For the upper bound of the proof theoretic strength we can rely on existing proof theoretic analysis of classical theories. We will interpret Martin-Löf Type Theory with W-type and one universe to Kripke Platek Set Theory with one inaccessible and finitely many admissibles above it.

Finally we investigate extensions of Martin-Löf Type Theory in order to get a higher proof theoretic strength. We look at the Mahlo Universe, and, if time permits, extensions beyond such as the autonomous Mahlo universe.