Three Lectures on Structural Proof Theory

3 – On Splitting and Cut Elimination in Deep Inference

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Course Notes

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Outline for Today

The Intuitive Idea of Splitting
Cut Elimination via Splitting
On Design: A First Order Proof System KSgr
Dealing with Quantifiers, in view of Splitting
On Herbrand Theorem
Some Thoughts about Analyticity
The Intuitive Idea of a Quasi-Polynomial Time Cut Elimination

Idea of Splitting

$$\wedge\,\dfrac{\overset{\Pi_1}{\vdash \Gamma, A} \qquad \overset{\Pi_2}{\vdash \Gamma, B}}{\vdash \Gamma, A \wedge B} \quad \leftrightsquigarrow \quad \begin{array}{c} \Pi \Big\Vert \\ K\{(A \wedge B)\} \end{array} \quad \Longrightarrow \exists \text{ proofs s.t. } \begin{array}{c} \Pi_1 \Big\Vert \\ K_\Gamma\{A\} \end{array} \text{ and } \begin{array}{c} \Pi_2 \Big\Vert \\ K_\Gamma\{B\} \end{array}$$

$$\Longrightarrow \exists \quad \begin{array}{c} C \vee \{\ \} \\ \Delta \Big\Vert \\ K\{\ \} \end{array}, \quad \begin{array}{c} \Pi_1 \Big\Vert \\ A \vee C \end{array} \text{ and } \begin{array}{c} \Pi_2 \Big\Vert \\ B \vee C \end{array}$$

$$\mathsf{Cut}\,\dfrac{\overset{\Pi_1'}{\vdash \Gamma, A} \qquad \overset{\Pi_2'}{\vdash \Gamma, \bar{A}}}{\vdash \Gamma} \quad \leftrightsquigarrow \quad \begin{array}{c} \Pi \Big\Vert \\ \mathsf{i}{\uparrow}\,\dfrac{K\{(a \wedge \bar{a})\}}{K\{\mathsf{f}\}} \end{array} \quad \Longrightarrow \text{ there exist two proofs } \begin{array}{c} \Pi_1 \Big\Vert \\ K\{a\} \end{array} \text{ and } \begin{array}{c} \Pi_2 \Big\Vert \\ K\{\bar{a}\} \end{array}$$

or, equivalently, two derivations $\begin{array}{c} a \\ \Delta_1 \Big\Vert \\ K\{\mathsf{f}\} \end{array}$ $\begin{array}{c} \bar{a} \\ \Delta_2 \Big\Vert \\ K\{\mathsf{f}\} \end{array}$ s.t.

$$\mathsf{c}{\downarrow}\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{\mathsf{t}}{\begin{array}{ccc} a & & \bar{a} \\ \Delta_1 \Big\Vert & & \Delta_2 \Big\Vert \\ K\{\mathsf{f}\} & \vee & K\{\mathsf{f}\} \end{array}}}{K\{\mathsf{f}\}}$$

(everything is in the down fragment)

Cut Elimination via Splitting

- Atomic cut and locality in rules are key-features not available in sequent calculus, in general;
- Atomicity and deep inference are key elements that allow to re-unite branches of the sequent proof-tree (and this was used already in the propositional proof);
- Splitting theorems are common to several different logics and work under the same intuition: method proper of deep inference (for example [12, 14, 19, 1, 4])
- Extends to first order classical logic (differently from previously shown proof).
- Avoiding variable capture requires careful attention in splitting, influencing the design of rules for quantifiers.

We will see the proof of splitting in detail, based on Brünnler's [3] and already anticipated by Guglielmi.

The Proof System *KSgr*

Structures (*a* positive/negative atom)

$$S ::= \mathsf{f} \mid \mathsf{t} \mid a \mid [\, S, S \,] \mid (\, S, S \,) \mid \exists x S \mid \forall x S$$

Syntactic equivalence: AC for $\wedge/\vee$, renaming of bound variables (to avoid variable capture), De Morgan and units extended to quantifiers:

$$\bar{\mathsf{f}} = \mathsf{t} \qquad \overline{[R,T]} = (\bar{R}, \bar{T}) \qquad \overline{\exists x R} = \forall x \bar{R} \qquad \overline{\overline{p(\vec{\tau})}} = p(\vec{\tau})$$

$$\bar{\mathsf{t}} = \mathsf{f} \qquad \overline{(R,T)} = [\bar{R}, \bar{T}] \qquad \overline{\forall x R} = \exists x \bar{R}$$

$$[R, \mathsf{f}] = R \qquad [\mathsf{t}, \mathsf{t}] = \mathsf{t} \qquad \exists x \mathsf{f} = \mathsf{f} = \forall x \mathsf{f}$$

$$(R, \mathsf{t}) = R \qquad (\mathsf{f}, \mathsf{f}) = \mathsf{f} \qquad \forall x \mathsf{t} = \mathsf{t} = \exists x \mathsf{t}$$

KSgr extends *KSg* with instantiation $n{\downarrow}$ and retraction $r{\downarrow}$ (instead of $u{\downarrow}$):

- $P\{\ \}$ is a propositional context, so x does not occur therein
- $n{\downarrow}$ substitution is capture-avoiding

$$\mathsf{i}{\downarrow}\,\frac{S\{\mathsf{t}\}}{S[R, \bar{R}]} \qquad \mathsf{w}{\downarrow}\,\frac{S\{\mathsf{f}\}}{S\{R\}} \qquad \mathsf{c}{\downarrow}\,\frac{S[R, R]}{S\{R\}}$$

$$\mathsf{s}\,\frac{S([R,T],U)}{S[(R,U),T]} \qquad \mathsf{r}{\downarrow}\,\frac{S\{\forall x P\{R\}\}}{S\{P\{\forall x R\}\}} \qquad \mathsf{n}{\downarrow}\,\frac{S\{R[x/t]\}}{S\{\exists x R\}}$$

$$\mathsf{u}{\downarrow}\,\frac{S\{\forall x [R, T]\}}{S[\forall x R, \exists x T]}$$

The Proof System *KSgr* - cont'd

- We can safely assume that instances of $i\downarrow$ and $w\downarrow$ are atomic (they are derivable):

PROPOSITION 2.12. *The rules* $\mathsf{i}\downarrow$ *and* $\mathsf{w}\downarrow$ *are derivable for* $\{\mathsf{ai}\downarrow, \mathsf{s}, \mathsf{r}\downarrow, \mathsf{n}\downarrow\}$ *and* $\{\mathsf{aw}\downarrow, \mathsf{s}\}$, *respectively. Dually, the rules* $\mathsf{i}\uparrow$ *and* $\mathsf{w}\uparrow$ *are derivable for* $\{\mathsf{ai}\uparrow, \mathsf{s}, \mathsf{r}\uparrow, \mathsf{n}\uparrow\}$ *and* $\{\mathsf{aw}\uparrow, \mathsf{s}\}$, *respectively.*

- *KSgr* is equivalent to *SKSgr*;
- an indirect proof of cut-elimination (through sequent calculus) exists.

Quantifiers and Splitting

Consider this:

$$\text{Cut}\ \frac{\begin{array}{c}\Pi_1\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\Pi_2\\ \vdash \Gamma, \bar{A}\end{array}}{\vdash \Gamma} \quad \rightsquigarrow \quad \begin{array}{c} {\scriptstyle(\Pi_1,\Pi_2)} \Big\| \\ \mathsf{s}^2\ \dfrac{([\Gamma, A], [\Gamma, \bar{A}])}{\mathsf{i}\!\uparrow\ \dfrac{[\Gamma, \Gamma, (A, \bar{A})]}{\mathsf{c}\!\downarrow\ \dfrac{[\Gamma, \Gamma]}{\Gamma}}} \end{array}$$

- *i*↑ does not 'split the requirements' arising from quantifiers.
- *i*↑ can introduce a cut-formulae deeply and variables may be captured by quantifiers in the context;
- Solution – control the 'possibly offensive' existential quantifiers for the context, by using bigger cuts.
- Introduce the notions of splittable cut and solid cut rule.

Replacing Cuts with Others (Fit for Purpose)

Up-rules may be derived using 'bigger cuts' with the variant *si*↑

$$\rho\uparrow\,\frac{S\{T\}}{S\{R\}} \qquad \rightsquigarrow \qquad \mathsf{si}\uparrow\,\frac{\rho\downarrow\,\frac{\mathsf{s}\,\frac{\mathsf{i}\downarrow\,\frac{=\,\frac{S\{T\}}{(S\{T\},\mathsf{t})}}{(S\{T\},[S\{R\},\bar{S}\{\bar{R}\}])}}{[S\{R\},(S\{T\},\bar{S}\{\bar{R}\})]}}{[S\{R\},(S\{T\},\bar{S}\{\bar{T}\})]}}{S\{R\}}$$

- Splittable cut *si*↑ is a cut inside a splittable context $S\{\ \}$, i.e. the hole is not in the scope of an existential qtf.
- If the cut-formula is a quantifier or an atom, we call the cut solid.

$$\textit{For each proof}\ \begin{array}{c}\|\ \mathsf{SKSgr}\\ T\end{array}\ \textit{there is a proof}\ \begin{array}{c}\|\ \mathsf{KSgr}\cup\{\mathsf{si}\uparrow\}\\ T\end{array}.$$

Sketch of the Cut Elimination Proof

For each proof $\overset{}{\underset{T}{\|}}{}^{\mathsf{KSgr}\cup\{\mathsf{si}\uparrow\}}$ there exists a proof $\overset{}{\underset{T}{\|}}{}^{\mathsf{KSgr}}$.

1. (Splittable) cuts are replaced by (splittable) solid cuts (induction on cut rank)
   - Solid cuts involve atomic and (existential) quantified formulae;
   - Existential cuts are handled first.
     The topmost one is transformed by cut-reduction (induction on maximal cut-rank). This reduces the proof above it to one whose cut rank is at most 1.
2. Once that all cuts are in atomic form, eliminate atomic cuts.

Replacing Cuts – cont'd

How to enforce the use of solid splittable cuts?

- By "guarding" the applicability of splittable cuts: unwanted existentials in the context should not enter the scope of the cut.
- The "guard" is related to the cut-rank of the cut-formula/derivation, a measure that counts the nested quantifiers of the cut-formula(e).
- $\mathit{si}_r\!\uparrow$ proviso – cut-rank at most r ($r \geq 0$).

$$
\mathsf{si}_r\!\uparrow \frac{S(\bar{R},\bar{T},[R,T])}{S\{\mathsf{f}\}}
\quad\rightsquigarrow\quad
\mathsf{si}_r\!\uparrow \frac{\mathsf{si}_r\!\uparrow \dfrac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{S(\bar{R},\bar{T},[R,T])}{S(\bar{R},[(\bar{T},T),R])}}{S[(\bar{T},T),(\bar{R},R)]}}{S(\bar{R},R)}}{S\{\mathsf{f}\}}
$$

(The instance on the left is not solid).
$\mathit{si}_r\!\uparrow$ is derivable for s and for solid $\mathit{si}_r\!\uparrow$ (i.e. atomic or qtf).

Replacing Cuts – cont'd

Are the new cuts 'safe' to be used?

1. Up-rules that were admissible with usual cuts, are still admissible in a system with *si* $\uparrow$ (already seen), but check that
2. the transformation does not interfere with the cut-rank (increase in cut-rank or length of proof)

For example, in proof Π (below left):
– we can 'pull' t upwards through all *a*'s we may meet, and in any place (context /redex/contractum), this happens in other rules..
– .. apart in interaction (right), where it will vanish:

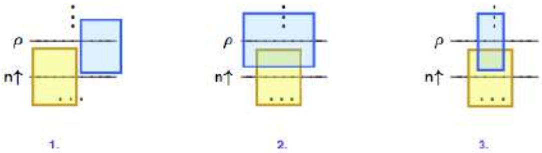
$$\mathsf{aw}\uparrow \frac{\begin{array}{c}\Pi \Big\| \mathsf{KSgr} \cup \{\mathsf{si}\uparrow\} \\ T\{a\}\end{array}}{T\{\mathsf{t}\}} \qquad\qquad \mathsf{ai}\downarrow \frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \quad \leadsto \quad \mathsf{aw}\downarrow \frac{= \dfrac{S\{\mathsf{t}\}}{S[\mathsf{t},\mathsf{f}]}}{S[\mathsf{t},\bar{a}]}$$

Technical – permutability details

More elaborate is $n\uparrow$: case analysis for rule permutability

1. Contractum of $n\uparrow$ inside the context of ρ: permute rules
2. Contractum of $n\uparrow$ inside a formula of redex of ρ ($s, c\downarrow, r\downarrow, n\downarrow$)
3. redex of ρ inside contractum of $n\uparrow$

$$\mathsf{n}\uparrow\,\dfrac{\rho\,\dfrac{\overset{\Vert\,\mathsf{KSgr}\cup\{\mathsf{si}\uparrow\}}{M}}{S\{\forall x R\}}}{S\{R\{x/\tau\}\}}$$



Technical – permutability cont'd

For example, these are cases in situation 2., when ρ is not $n\!\uparrow$:

$$\begin{array}{rc}
 & S''\{[S'\{\forall x R\}\ S'\{\forall x R\}]\} \\ \cline{2-2}
\mathsf{c}{\downarrow} & S''\{S'\{\forall x R\}\} \\ \cline{2-2}
\mathsf{n}{\uparrow} & S''\{S'\{R\{x/\tau\}\}\}
\end{array}
\quad\leadsto\quad
\begin{array}{rc}
 & S''\{[S'\{\forall x R\}\ S'\{\forall x R\}]\} \\ \cline{2-2}
\mathsf{n}{\uparrow} & S''\{[S'\{R\{x/\tau\}\}\ S'\{\forall x R\}]\} \\ \cline{2-2}
\mathsf{n}{\uparrow} & S''\{[S'\{R\{x/\tau\}\}\ S'\{R\{x/\tau\}\}]\} \\ \cline{2-2}
\mathsf{c}{\downarrow} & S''\{S'\{R\{x/\tau\}\}\}
\end{array}$$

$$\begin{array}{rc}
 & S''\{([S'\{\forall x R\}\ N]\ \ M)\} \\ \cline{2-2}
\mathsf{s} & S''\{[(S'\{\forall x R\}\ M)\ \ N]\} \\ \cline{2-2}
\mathsf{n}{\uparrow} & S''\{[(S'\{R\{x/\tau\}\}\ M)\ \ N]\}
\end{array}
\quad\leadsto\quad
\begin{array}{rc}
 & S''\{([S'\{\forall x R\}\ N]\ \ M)\} \\ \cline{2-2}
\mathsf{n}{\uparrow} & S''\{([S'\{R\{x/\tau\}\}\ N]\ \ M)\} \\ \cline{2-2}
\mathsf{s} & S''\{[(S'\{R\{x/\tau\}\}\ M)\ \ N]\}
\end{array}$$

$$\begin{array}{rc}
 & S\{\forall y\ P\{S''\{S'\{\forall x R\}\}\ \}\} \\ \cline{2-2}
\mathsf{r}{\downarrow} & S\{P\{\forall y\ S''\{S'\{\forall x R\}\}\ \}\} \\ \cline{2-2}
\mathsf{n}{\uparrow} & S\{P\{\forall y\ S''\{S'\{R\{x/\tau\}\}\}\ \}\}
\end{array}
\quad\leadsto\quad
\begin{array}{rc}
 & S\{\forall y\ P\{S''\{S'\{\forall x R\}\}\ \}\} \\ \cline{2-2}
\mathsf{n}{\uparrow} & S\{\forall y\ P\{\ S''\{S'\{R\{x/\tau\}\}\}\ \}\} \\ \cline{2-2}
\mathsf{r}{\downarrow} & S\{P\{\forall y\ S''\{S'\{R\{x/\tau\}\}\}\ \}\}
\end{array}$$

Technical – permutability cont'd

In case 2. when ρ is $n\uparrow$:

$$
\mathsf{n}\!\uparrow \frac{\mathsf{n}\!\downarrow \dfrac{S\{R\{\forall y T\}[x/\tau_1]\}}{S\{\exists x R\{\forall y T\}\}}}{S\{\exists x R\{T[y/\tau_2]\}\}}
\quad \rightsquigarrow \quad
\mathsf{n}\!\downarrow \frac{= \dfrac{\mathsf{n}\!\uparrow \dfrac{= \dfrac{S\{R\{\forall y T\}[x/\tau_1]\}}{S\{R[x/\tau_1]\{\forall y T[x/\tau_1]\}\}}}{S\{R[x/\tau_1]\{T[x/\tau_1][y/\tau_2[x/\tau_1]]\}\}}}{S\{R\{T[y/\tau_2]\}[x/\tau_1]\}}}{S\{\exists x R\{T[y/\tau_2]\}\}}
$$

Assume variables are named apart;

1. no var in τ_1 occurs bound in $R\{\forall y T\}$ (lilac);
2. no var in τ_2 occurs bound in T (green);

Technical – permutability cont'd

In case 2. when ρ is $n\uparrow$:

$$\dfrac{\dfrac{S\{R\{\forall y T\}[x/\tau_1]\}}{S\{\exists x R\{\forall y T\}\}}\;\mathsf{n}\!\downarrow}{S\{\exists x R\{T[y/\tau_2]\}\}}\;\mathsf{n}\!\uparrow \quad\leadsto\quad \dfrac{\dfrac{\dfrac{\dfrac{S\{R\{\forall y T\}[x/\tau_1]\}}{S\{R[x/\tau_1]\{\forall y T[x/\tau_1]\}\}}=}{S\{R[x/\tau_1]\{T[x/\tau_1][y/\tau_2[x/\tau_1]]\}\}}\;\mathsf{n}\!\uparrow}{S\{R\{T[y/\tau_2]\}[x/\tau_1]\}}=}{S\{\exists x R\{T[y/\tau_2]\}\}}\;\mathsf{n}\!\downarrow$$

Assume variables are named apart;

1. no var in τ_1 occurs bound in $R\{\forall y T\}$ (lilac);
2. no var in τ_2 occurs bound in T (green);

   Hence, the topmost $=$ is sound ($[x/\tau_1]$ can be distributed), and $\tau_2[x/\tau_1]$ is free for y in $T[x/\tau_1]$, so $n\uparrow$ is sound; τ_1 is free for x in $R\{T[y/\tau_2]\}$, making the lowermost $=$ and $n\downarrow$ both sound.

Technical – permutability cont'd

In case 2. when ρ is $n\uparrow$:

$$
\dfrac{\dfrac{S\{R\{\forall y T\}[x/\tau_1]\}}{S\{\exists x R\{\forall y T\}\}}\,\mathsf{n}\!\downarrow}{S\{\exists x R\{T[y/\tau_2]\}\}}\,\mathsf{n}\!\uparrow
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{S\{R\{\forall y T\}[x/\tau_1]\}}{S\{R[x/\tau_1]\{\forall y T[x/\tau_1]\}\}}\,=}{S\{R[x/\tau_1]\{T[x/\tau_1][y/\tau_2[x/\tau_1]]\}\}}\,\mathsf{n}\!\uparrow}{\dfrac{S\{R\{T[y/\tau_2]\}[x/\tau_1]\}}{S\{\exists x R\{T[y/\tau_2]\}\}}\,\mathsf{n}\!\downarrow}\,=
$$

Assume variables are named apart;

1. no var in τ_1 occurs bound in $R\{\forall y T\}$ (lilac);
2. no var in τ_2 occurs bound in T (green);

   Hence, the topmost $=$ is sound ($[x/\tau_1]$ can be distributed), and $\tau_2[x/\tau_1]$ is free for y in $T[x/\tau_1]$, so $n\uparrow$ is sound; τ_1 is free for x in $R\{T[y/\tau_2]\}$, making the lowermost $=$ and $n\downarrow$ both sound.

Technical – permutability cont'd

In case 3. permutability may require some renaming of bound variables (for example when ρ is $r\!\downarrow$ or $n\!\uparrow$).

The only critical case is the overlapping on an universal quantifier:

$$\mathsf{n}\!\uparrow \frac{\mathsf{r}\!\downarrow \dfrac{S\{\forall x P\{R\}\}}{S\{P\{\forall x R\}\}}}{S\{P\{R[x/\tau]\}\}} \quad \rightsquigarrow \quad \mathsf{n}\!\uparrow \frac{S\{\forall x P\{R\}\}}{S\{P\{R[x/\tau]\}\}}$$

'Splitting' the Context

- By permutability, eventually $w\uparrow$ and $n\uparrow$ can be eliminated, in presence of splittable cuts, with no impact on the cut rank.
- Hence, we just have to deal with $KSgr \cup \{si\uparrow\}$.

LEMMA 3.7 (Splitting). *Let $S\{\ \}$ be a splittable context and let $\forall\vec{x}$ be the sequence of all its universal quantifiers that have the hole in their scope.*

Then for each proof $\begin{array}{c}\Pi \| \mathsf{KSgr} \cup \{\mathsf{si}\uparrow\} \\ S(R,T)\end{array}$ *there are a formula U and proofs* $\begin{array}{c}\| \mathsf{KSgr} \cup \{\mathsf{si}\uparrow\} \\ [U,R]\end{array}$ *and* $\begin{array}{c}\| \mathsf{KSgr} \cup \{\mathsf{si}\uparrow\} \\ [U,T]\end{array}$ *and a derivation* $\begin{array}{c}\forall\vec{x}U \\ \| \{\mathsf{r}\downarrow\} \\ S\{\mathsf{f}\}\end{array}$ *such that the cut ranks of both proofs are smaller than or equal to the cut rank of Π.*

’Splitting’ the Context

- In the proof, attention goes to $\forall$ quantifiers, and in handling those splittable context $S\{\ \}$ whose hole is in the scope of a universal quantifier
- Some cases are below

$$
\begin{array}{c}
\Vert\ \mathsf{KSgr}\cup\{\mathsf{si}\uparrow\} \\
\mathsf{w}\uparrow\ \dfrac{S(R,T)}{S\{R\}} \\
\mathsf{n}\uparrow\ \dfrac{\phantom{S\{R\}}}{S'\{R\}} \\
\mathsf{s}^*\ \dfrac{\phantom{S'\{R\}}}{[S'\{\mathsf{f}\},R]}
\end{array}
\quad , \quad
\begin{array}{c}
\Vert\ \mathsf{KSgr}\cup\{\mathsf{si}\uparrow\} \\
\mathsf{w}\uparrow\ \dfrac{S(R,T)}{S\{T\}} \\
\mathsf{n}\uparrow\ \dfrac{\phantom{S\{T\}}}{S'\{T\}} \\
\mathsf{s}^*\ \dfrac{\phantom{S'\{T\}}}{[S'\{\mathsf{f}\},T]}
\end{array}
\quad , \quad
\begin{array}{c}
\forall\vec{x}S'\{\mathsf{f}\} \\
\Vert\ \{\mathsf{r}\downarrow\} \\
S\{\mathsf{f}\}
\end{array}
$$

Atomic Cut Elimination

$$\textit{For each proof}\quad \mathsf{si}\!\uparrow \frac{\begin{array}{c}\Pi \,\Big\|\, \mathsf{KSgr} \\ T(a,\bar{a})\end{array}}{T\{\mathsf{f}\}} \quad \textit{there is a proof} \quad \begin{array}{c}\Big\|\, \mathsf{KSgr} \\ T\{\mathsf{f}\}\end{array} .$$

Sketch of proof

1. - Apply splitting on Π (Π_2 is a proof from *a* to *U*), to obtain

$$\begin{array}{c}\Pi_1 \,\Big\|\, \mathsf{KSgr} \\ [U,a]\end{array} \;,\quad \begin{array}{c}\Pi_2 \,\Big\|\, \mathsf{KSgr} \\ [U,\bar{a}]\end{array} \quad \text{and} \quad \begin{array}{c}\forall\vec{x}U \\ \Delta \,\Big\|\, \{\mathsf{r}\!\downarrow\} \\ T\{\mathsf{f}\}\end{array}$$

2. - Bottom-up in Π_1, replace a/U. Renaming in $r\!\downarrow$ ($n\downarrow$ is absent). Transform $ai\!\downarrow$ (left), combine in final proof (box)

$$\mathsf{ai}\!\downarrow \frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \quad \leadsto \quad \begin{array}{c}S\{\mathsf{t}\} \\ S\{\Pi_2\} \,\Big\|\, \mathsf{KSgr} \\ S[U,\bar{a}]\end{array}$$

$$\boxed{\begin{array}{c}\forall\vec{x}\,\Pi_3 \,\Big\|\, \mathsf{KSgr} \\ \mathsf{c}\!\downarrow \dfrac{\forall\vec{x}[U,U]}{\forall\vec{x}U} \\ \Delta \,\Big\|\, \{\mathsf{r}\!\downarrow\} \\ T\{\mathsf{f}\}\end{array}}$$

Cut Reduction (Existential Formulae)

Cut formula has form $\exists x R$ – inference rules may be applied inside R.

Generalise the technique to the case of n:

- n-context – a formula with n holes $\{\ \}$
- splittable n-context – no hole is in the scope of an existential quantifier.
- Given a proof Π of $[U\ \forall x R]$ in $KSgr \cup \{si\uparrow\}$, and $n \geq 1$, define

$$\text{plug}_{\Pi,n}\ \frac{S\{\exists x R_1\}\ldots\{\exists x R_n\}}{S\{U\}\ldots\{U\}}$$

  where $S\{\ \}\ldots\{\ \}$ is splittable, and cut-free Δ_i from R_i to $\bar{R}$ exist.
- As in the atomic case, splitting lemma is applied; the parametric plug rule applied to each existential, in parallel.

Details of Cut Reduction (Existential Formulae)

Statement

For each proof

$$\mathsf{si}_{r+1}\!\uparrow \frac{\overset{\Pi \,\Big\| \,\mathsf{KSgr} \cup \{\mathsf{si}_r\uparrow\}}{T(\forall x R, \exists x \bar{R})}}{T\{\mathsf{f}\}}$$

there is a proof

$$\overset{\Big\| \,\mathsf{KSgr} \cup \{\mathsf{si}_r\uparrow\}}{T\{\mathsf{f}\}} .$$

Sketch of proof

1. Apply splitting

$$\overset{\Pi_1 \,\Big\| \,\mathsf{KSgr} \cup \{\mathsf{si}_r\uparrow\}}{[U, \exists x \bar{R}]} , \quad \overset{\Pi_2 \,\Big\| \,\mathsf{KSgr} \cup \{\mathsf{si}_r\uparrow\}}{[U, \forall x R]} \quad \text{and} \quad \begin{array}{c} \forall \vec{x} U \\ \Delta \,\Big\| \,\{\mathsf{r}\downarrow\} \\ T\{\mathsf{f}\} \end{array}$$

2. Plug proof 2 into proof 1. The plug is then pushed up and let disappear

$$\begin{array}{c} \forall \vec{x}\Pi_1 \,\Big\| \,\mathsf{KSgr} \cup \{\mathsf{si}_r\uparrow\} \\ \mathsf{plug}_{\Pi_2,1} \dfrac{\forall \vec{x} [U, \exists x \bar{R}]}{\mathsf{c}\!\downarrow \dfrac{\forall \vec{x} [U, U]}{\forall \vec{x} U}} \\ \Delta \,\Big\| \,\{\mathsf{r}\downarrow\} \\ T\{\mathsf{f}\} \end{array}$$

Details of Cut Reduction (Existential Formulae)

$$\mathsf{plug}_{\Pi_2,n}\dfrac{\mathsf{n}\!\downarrow\dfrac{S'\{R_i[x/\tau]\}}{S'\{\exists x R_i\}}}{S\{U\}} \quad \leadsto \quad \mathsf{si}_r\!\uparrow\dfrac{\mathsf{s}\dfrac{\begin{array}{c}\mathsf{plug}_{\Pi_2,n-1}\dfrac{S'\{R_i[x/\tau]\}}{S\{R_i[x/\tau]\}} \\ S(\Delta_i[x/\tau],\Pi_2') \Big\| \mathsf{KSgr}\cup\{\mathsf{si}_r\!\uparrow\} \\ S(\bar{R}[x/\tau],[U,R[x/\tau]])\end{array}}{S[U,(\bar{R}[x/\tau],R[x/\tau])]}}{S\{U\}} \quad ,$$

3. Rule above plug is applied inside R_i: plug is added to the corresponding Delta. In this specific case the derivation is instantiated

4. Termination when plug reaches the top:

$$\mathsf{w}\!\downarrow^n \dfrac{S\{\mathsf{f}\}\ldots\{\mathsf{f}\}}{S\{U\}\ldots\{U\}} \qquad \text{and} \qquad \begin{array}{c} \forall\vec{x}\,(\Pi_2'',\Delta_i) \Big\| \\ \mathsf{si}_r\!\uparrow\dfrac{\mathsf{s}\dfrac{\forall\vec{x}([U,R],\bar{R})}{\forall\vec{x}[U,(R,\bar{R})]}}{\forall\vec{x}U} \\ \Delta \Big\| \\ T\{\mathsf{f}\}\end{array}$$

all R_i are false (left), at least one is true (right)

Herbrand Theorem and Cut Elimination

There are two forms of Herbrand theorem:

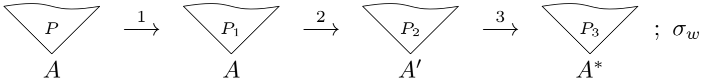
- Weak form: F quantifier free.

  $$\exists x_1..\exists x_n F \text{ valid } \Longrightarrow \exists t_{ij} : \vdash F(t_{1,1}, \ldots, t_{1,k_1}) \vee \cdots \vee F(t_{n,1}, \ldots, t_{n,k_n})$$

  (in particular, we look for a cut-free proof)
- Strong form: A f.o.f A is valid iff it has a Herbrand proof.
  A Herbrand proof of A consists of a prenexification A^* of a strong $\vee$-expansion
- We focus on the strong form, and compare the situation in the sequent calculus and in deep inference proof systems.

Herbrand Theorem and Cut Elimination

A is valid iff it has a Herbrand proof. – In sequent calculus:



P cut-free; A any formula.

1. P_1 cut-free, with contraction on propositional and existential formulae only.
2. P_2 cut-free. A' is strong $\vee$ expansion of A (i.e. saturated with $\exists$-formulae). Contraction on $\exists$-formula is replaced with $\vee_R$, and the change propagated in the proof. After this step only contraction on propositions.
3. Prenexification. Pulling quantifiers to the front causes quantifiers rules to go downwards in the proof. P_3 is the Herbrand proof; its instances of $\exists_R$ give the terms for σ_w.

Herbrand Theorem – cont'd

In deep inference and using the system *SKSgr* we would need to

1. decompose contraction (atomic, first order case, already seen)

$$\text{ac}\!\downarrow \frac{S[a,a]}{S\{a\}} \qquad \text{m}\, \frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

$$\text{qc}\!\downarrow \frac{S[\exists x R, \exists x R]}{S\{\exists x R\}} \qquad \text{m}_2\!\downarrow \frac{S[\forall x R, \forall x T]}{S\{\forall x [R,T]\}}$$

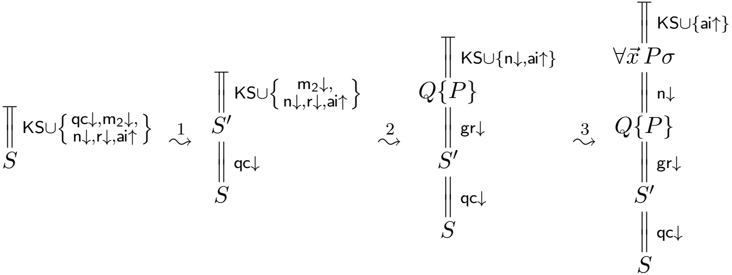
2. handle prenexification, and the retract $r\downarrow$ rule would need to be extended

$$\text{gr}\!\downarrow \frac{S\{Q\{P\{R\}\}\}}{S\{P\{Q\{R\}\}\}}$$

where $Q\{\ \}$ sequence of quantifiers, $P\{\ \}$ propositional context.
No variable in $P\{\ \}$ is bound by a quantifier in $Q\{\ \}$ in the premise.

Herbrand Theorem – cont'd

Each proof in *SKSgr* has one with the shape on the right, for some substitution σ, propositional formula *P* and context $Q\{\ \}$ of quantifiers.



0. – From the proof in *SKSgr* obtain the leftmost one, using the transformations with splittable cuts. Decompose contraction.

1. – Move downwards contractions on quantifiers.

2. – Factorise *S′*. Construct the prenex normal form $Q\{P\}$; its proof above contains prenex formulae only.

3. – Separate instantiations. The remaining topmost proof is propositional (atomic cuts only).

Herbrand Theorem – cont'd

However,

- Proof systems could be specifically designed so to better support our ability to extract Herbrand proofs.
- Ben Ralph proposes two improvements to *SKSgr* resulting in two different proof systems: the resulting Herbrand proof has a different structure.
- The improvements stem from a need to simplify the context management, and to facilitate technical aspects in dealing with substitution into the D.I. formalism of open deduction [13, 15].
- We just mention the scope of these (recent) works. A compact reference is [17], full details in [18]

Herbrand Theorem – cont'd

- System *KSh*1: $r_i\downarrow$ rules (*B* free for *x*) simulate $gr\downarrow$ and make prenexification easier.

$$
\mathsf{KSh1} = \mathsf{KS} + \boxed{
\begin{array}{cccccc}
\mathsf{r1}\!\downarrow \dfrac{\forall x[A \vee B]}{[\forall x A \vee B]} & \mathsf{r2}\!\downarrow \dfrac{\forall x(A \wedge B)}{(\forall x A \wedge B)} & \mathsf{n}\!\downarrow \dfrac{A\{x \Leftarrow t\}}{\exists x A} \\[2ex]
\mathsf{r3}\!\downarrow \dfrac{\exists x[A \vee B]}{[\exists x A \vee B]} & \mathsf{r4}\!\downarrow \dfrac{\exists x(A \wedge B)}{(\exists x A \wedge B)} & \mathsf{qc}\!\downarrow \dfrac{\exists x A \vee \exists x A}{\exists x A}
\end{array}}
+
\boxed{
\begin{array}{rclrcl}
\forall x A & = & \forall z A\{x \Leftarrow z\} & \exists z A & = & \exists z A\{x \Leftarrow z\} \\
\forall x \forall y A & = & \forall y \forall x A & \exists x \exists y A & = & \exists y \exists x A \\
\forall x \mathsf{t} & = & \mathsf{t} = \exists x \mathsf{t} & \forall x \mathsf{f} & = & \mathsf{f} = \exists x \mathsf{f}
\end{array}}
$$

$$
\begin{array}{c}
\Vert\, \mathsf{KS} \\
\forall x B\{y \Leftarrow t\} \\
\Vert\, \{\mathsf{n}\!\downarrow\} \\
Q\{B\} \\
\Vert\, \{\mathsf{r1}\!\downarrow, \mathsf{r2}\!\downarrow, \mathsf{r3}\!\downarrow, \mathsf{r4}\!\downarrow\} \\
A' \\
\Vert\, \{\mathsf{qc}\!\downarrow\} \\
A
\end{array}
$$

- Every proof in *KSh*1 can be converted to a Herbrand proof.

Herbrand Theorem – cont'd

- System *KSh2*: $h\downarrow$ (Herbrand expander) and $\exists w \downarrow$ (existential weakening) further help lifting to the level of the formalism the handling of substitutions.
- This allows to tighten the relation between open deduction proofs and expansion proof, providing a normal form of Herbrand proofs.

$$
\mathsf{KSh2} = \mathsf{KS} + \boxed{\begin{array}{cc}
\mathsf{r1}\!\downarrow \dfrac{\forall x[A \vee B]}{[\forall x A \vee B]} & \mathsf{h}\!\downarrow \dfrac{\exists x A \vee A\{x \Leftarrow t\}}{\exists x A} \\[2ex]
\mathsf{r2}\!\downarrow \dfrac{\forall x(A \wedge B)}{(\forall x A \wedge B)} & \exists\mathsf{w}\!\downarrow \dfrac{\mathsf{f}}{\exists x A}
\end{array}}
$$

$$
+
$$

$$
\boxed{\begin{array}{rclrcl}
\forall x A & = & \forall z A\{x \Leftarrow z\} & \exists z A & = & \exists z A\{x \Leftarrow z\} \\
\forall x \forall y A & = & \forall y \forall x A & \exists x \exists y A & = & \exists y \exists x A \\
\forall x \mathsf{t} & = & \mathsf{t} = \exists x \mathsf{t} & \forall x \mathsf{f} & = & \mathsf{f} = \exists x \mathsf{f}
\end{array}}
$$

Herbrand's Thm - cont'd

To recap:

- Sequent calculus does not allow to represent, in the proof of the original formula, neither the expansion nor the prenexification: the reason is that the rules work on the main connective.
- In contrast, in deep inference, these phases can be integrated in the proof very smoothly.
- Herbrand Theorem and Splitting Theorem for Cut elimination (scalable to predicate logic), both formulated entirely inside deep inference, give to the deep inference methodology a proper status.

Some Proposed Activities

References for this part are Kai Brünnler's [3] and Sam Buss' [9], adn Ben Ralph's [17] (for Herbrand's Theorems). Be aware that Kai reverses the use of the wording 'redex and contractum' of a rule, in that paper!

- One might want to try and complete the study of permutability of some pair rules, for example $n\!\uparrow -r\!\downarrow$ or $n\!\uparrow -n\!\downarrow$ just to familiarize with the method. Details are on the technical descriptions in this group of slides.
- In the first batch of slides two sequent proofs (with cut and cut-free) of $\vdash \exists x.\forall y(p(x) \supset p(y))$ are given. Compare them with the proofs in deep inference of the corresponding formula in negation normal form $\vdash \exists x.\forall y(\bar{p}(x) \vee p(y)))$.
  One can then use them as guidance to perform a cut-elimination with splitting in deep inference, and compare and contrast with the situation in sequent calculus. The same, again, in relation to Herbrand theorem.

Analyticity - some Thoughts

Three properties in sequent calculus systems are considered akin

- the subformula property in rules;
- the system is cut-free;
- the system is analytic;

AND analytic proofs reduce non-determinism in proof search (also f.o.)

What would 'analyticity' in deep inference be alike, where:

- the same connectives compose formulae, as well as derivations / (rules applied at any depth);
- there is a duality between down- and up-rules, these latter ones derivable by cut;
- splitting theorem implies cut elimination
- .. just to mention a few.. ?

(A compact reference is [5])

Analyticity - some observations

(Naif) – "A rule would be analytic, if, given an instance of its conclusion, the set of possible instances of the premiss is finite"

To the effect that this *Finitary cut rule* would qualify as (naif)-analytic

$$K\left\{ \mathsf{fai}\uparrow \frac{p(\vec{x}) \wedge \bar{p}(\vec{x})}{\mathsf{f}} \right\} \quad \text{, where } p \text{ appears in } K\{\ \}.$$

- the premiss is finitely generating;
- *fai* $\uparrow$ can replace the general cut, at a polynomial cost in size of proofs, and
- via transformations that are local (in contrast to unbounded copying of chunks in the sequent calculus)

BUT reducing non-determinism cannot be so easy, we should rather exclude *fai* $\uparrow$ – stronger notion

Analyticity - some observations

(finitely generating rule + bounded generation driven by the conclusion)

Definition 1. For every formula B, context $K\{ \}$ and rule r, we define the set of *premisses of B in $K\{ \}$ via r*:

$$\mathsf{pr}(B, K\{ \}, r) = \left\{ A \;\middle|\; K\left\{ r\frac{A}{B} \right\} \right\} \quad .$$

Given a rule r:

1. if, for every B and $K\{ \}$, the set $\mathsf{pr}(B, K\{ \}, r)$ is finite, then we say that r is *finitely generating*;
2. if, for every B, there is a natural number n such that for every context $K\{ \}$ we have $|\mathsf{pr}(B, K\{ \}, r)| < n$, then we say that r is *analytic*.

To the effect that

- *fai* $\uparrow$ would not qualify as analytic;
- all the down fragment would

$$\mathsf{i}{\downarrow}\,\frac{\mathsf{t}}{A \vee \bar{A}} \qquad \mathsf{w}{\downarrow}\,\frac{\mathsf{f}}{A} \qquad \mathsf{c}{\downarrow}\,\frac{A \vee A}{A} \qquad \mathsf{s}\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$

- .. as well as some rules of the up-fragment: $c \uparrow$ (and other linear rules reshuffling information, in systems for other logics).

Analyticity - some observations

This notion better reflects 'common insights' from sequent calculi – Splitting theorems

- are formulated on the down fragment whose rules are 'analytic';
- imply cut elimination;
- inform proof search (by reducing the proof search space), addressing non-determinism

$c\uparrow$ is an extra asset towards complexity because

- it provides 'dagness' (sharing);
- it supports the construction of a quasi-polynomial ($n^{O(log n)}$) cut-elimination procedure for classical logic [8, 7] (related sources: [16], [2], [11])..
- *therefore*, $KSg \cup \{c\uparrow\}$ q.p. simulates $SKSg$.
- KSg (analytic) outperforms analytic sequent calculus on Statman's tautologies (exponential speed up). With cut, it poly-simulates Frege systems. $c\uparrow$ is used to show that just a limited depth is indeed necessary to reach bounded Frege systems [6, 10]

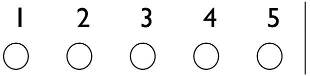
Sketch - Quasi-polynomial Time Cut-Elimination

This procedure uses Threshold Formulae

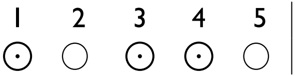
- They realise boolean threshold functions, i.e. boolean functions that are true iff at least k out of n inputs are true [20].
- Many different ways to encode them in a formula.
- Problem: find an encoding that allows us to formulate a certain theorem;
- The property stated by that theorem strongly depends on the proof system that we adopt!

We can simplify the definition of threshold formulae used by Atserias et al, to work on system *SKS*.

Threshold Functions - intuition



Threshold Functions - intuition



at least 3 out of 5 are true

Threshold Functions - intuition

1	2	3	4	5	
⊙	⊙	⊙	⊙	○	at least 3 out of 5 are true

Threshold Functions - intuition

1	2	3	4	5	
⊙	⊙	⊙	○	⊙	at least 3 out of 5 are true

Threshold Functions - intuition

1	2	3	4	5
$\odot$	$\bigcirc$	$\odot$	$\odot$	$\sqcup$

Set 5 to false.

At least 3 out of 5 are true;

At least 3 out of the remaining 4 are true;

Threshold Functions - intuition

1	2	3	4	5
⊙	○	⊙	⊙	⊔

Set 5 to false.

At least 3 out of 5 are true;

At least 3 out of the remaining 4 are true;

1	2	3	4	5
⊙	○	⊙	⊙	⊍

Set 5 to true.

At least 3 out of 5 are true;

At least 3 out of the remaining 4 are true;

At least 4 out 5 are true;

Threshold Functions - intuition

1	2	3	4	5
⊙	○	⊙	⊙	⊔

Set 5 to false.

At least 3 out of 5 are true;

At least 3 out of the remaining 4 are true;

1	2	3	4	5
⊙	○	⊙	⊙	⊔·

Set 5 to true.

At least 3 out of 5 are true;

At least 3 out of the remaining 4 are true;

At least 4 out 5 are true;

Intuitive use of Threshold Formulae (Splitting)

$$\mathsf{i}\!\uparrow \frac{\begin{array}{c}\Pi \Big\| \\ K\{(a\ \bar{a})\}\end{array}}{K\{\mathsf{f}\}} \implies \exists \Delta_1, \Delta_2 : \quad \begin{array}{c} a \\ \Delta_1 \Big\| \\ K\{\mathsf{f}\}\end{array} \qquad \begin{array}{c} \bar{a} \\ \Delta_2 \Big\| \\ K\{\mathsf{f}\}\end{array} \quad \text{such that} \quad \mathsf{c}\!\downarrow \frac{\mathsf{i}\!\downarrow \dfrac{\mathsf{t}}{\begin{array}{ccc} a & & \bar{a} \\ \Delta_1 \Big\| & & \Delta_2 \Big\| \\ K\{\mathsf{f}\} & \vee & K\{\mathsf{f}\}\end{array}}}{K\{\mathsf{f}\}}$$

$$= \mathsf{i}\!\uparrow \frac{\dfrac{K\{a \quad \vee \quad \{\tilde{a}^k\}\}}{\begin{array}{ccc} a & & \{\tilde{a}^k\} \\ \Delta_1' \Big\| & & \Delta_{QP} \Big\| \\ K\,\{a & \wedge & \{\tilde{a}^{k+1}\}\}\end{array}}}{K\{\mathsf{f}\}}$$

- $\tilde{a}^k$ pseudo-complement of a at slice k: it " behaves" as $\bar{a}$;
- Top $a \vee \mathsf{t}$, Bottom $a \wedge \mathsf{f}$
- $$\begin{array}{c} \tilde{a}^k \\ \Delta_{QP} \Big\| \\ \tilde{a}^{k+1} \end{array}$$ Size of threshold formulae: quasi-polynomial growth
  Size of Δ_{QP} dominated by threshold formulae – quasi-polynomial

Threshold Formulae - a Definition

Definition 6. *For every $n = 2^m$, with $m \geq 0$, and $k \geq 0$, we define the operator θ_k^n inductively as follows:*

$$\theta_k^n(a_1, \ldots, a_n) =$$
$$= \begin{cases} \mathsf{t} & \text{if } k = 0 \\ \mathsf{f} & \text{if } k > n \\ a_1 & \text{if } n = k = 1 \\ \bigvee_{\substack{i+j=k \\ 0 \leq i,j \leq n/2}} \left(\theta_i^{n/2}(a_1, \ldots, a_{n/2}) \wedge \theta_j^{n/2}(a_{n/2+1}, \ldots, a_n) \right) & \text{otherwise.} \end{cases}$$

For any n atoms $a_1, \ldots, a_n$, we call $\theta_k^n(a_1, \ldots, a_n)$ the threshold formula at level *k (*with respect to *$a_1, \ldots, a_n$).*

For any $n = s^m, m, k \geq 0$ the size of $\Theta_k^n(a_1, \ldots, a_k)$ has a quasi-polynomial bound in n

Threshold Formulae - cont'd

Some examples (any *n*):

$$\begin{aligned}
\theta_0^2(a,b) &\equiv \mathtt{t} \quad , \\
\theta_1^2(a,b) &\equiv (\theta_1^1(a) \wedge \theta_0^1(b)) \vee (\theta_0^1(a) \wedge \theta_1^1(b)) \equiv (a \wedge \mathtt{t}) \vee (\mathtt{t} \wedge b) \\
&= a \vee b \quad , \\
\theta_2^2(a,b) &\equiv \theta_1^1(a) \wedge \theta_1^1(b) \\
&\equiv a \wedge b \quad ,
\end{aligned}$$

$$\begin{aligned}
\theta_0^3(a,b,c) &\equiv \mathtt{t} \quad , \\
\theta_1^3(a,b,c) &\equiv (\theta_1^1(a) \wedge \theta_0^2(b,c)) \vee (\theta_0^1(a) \wedge \theta_1^2(b,c)) \equiv (a \wedge \mathtt{t}) \vee (\mathtt{t} \wedge [(b \wedge \mathtt{t}) \vee (\mathtt{t} \wedge c)]) \\
&= a \vee b \vee c \quad , \\
\theta_2^3(a,b,c) &\equiv (\theta_1^1(a) \wedge \theta_1^2(b,c)) \vee (\theta_0^1(a) \wedge \theta_2^2(b,c)) \\
&= (a \wedge [b \vee c]) \vee (b \wedge c) \quad , \\
\theta_3^3(a,b,c) &\equiv \theta_1^1(a) \wedge \theta_2^2(b,c) \equiv (a \wedge (b \wedge c)) \\
&= a \wedge b \wedge c \quad ,
\end{aligned}$$

Threshold Formulae - cont'd

A property of threshold functions/formulae, captured in *SKS* by a specific derivation:

Lemma 8. *For any $n = 2^m$, with $m \geq 0$, $k \geq 0$ and $1 \leq i \leq n$, there exists a derivation*

$$\Gamma_k^i = \begin{array}{c} \theta_k^n(a_1, \dots, a_n)\{a_i/\mathsf{f}\} \\ \| \{\mathsf{aw}\downarrow, \mathsf{aw}\uparrow\} \\ \theta_{k+1}^n(a_1, \dots, a_n)\{a_i/\mathsf{t}\} \end{array},$$

whose size has a quasipolynomial bound in n.

Remarks:

- Both premiss and conclusion of Γ_k^i are logically equivalent to $\theta_k^{n-1}(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$, pseudo-complement of a_i
- (proof by usual context extraction in D.I. and by structural induction on the threshold formulae).

Using Threshold Formulae

Use pseudo-complements of a_l, instead of $\bar{a}_l$, with increasing levels of k:

1. Make a disjunction between a_l and its pseudocomplement at level k; propagate this pseudocomplement across instances of $i\downarrow$(LEFT);
2. Increase the k-level (CENTRE);
3. For each instance of $i\uparrow$ collect the conjunction between a_l and its pseudocomplement at level $k+1$ (dual of 1, RIGHT):

$$\begin{array}{c} \theta^n_k \, \boldsymbol{a}^n_1 \\ \| \\ a_l \vee (\theta^n_k \, \boldsymbol{a}^n_1)\{a_l/\mathsf{f}\} \end{array} \qquad \begin{array}{c} (\theta^n_k \, \boldsymbol{a}^n_1)\{a_l/\mathsf{f}\} \\ \| \\ (\theta^n_{k+1} \, \boldsymbol{a}^n_1)\{a_l/\mathsf{t}\} \end{array} \qquad \begin{array}{c} a_l \wedge (\theta^n_{k+1} \, \boldsymbol{a}^n_1)\{a_l/\mathsf{t}\} \\ \| \\ \theta^n_{k+1} \, \boldsymbol{a}^n_1 \end{array}$$

- The derivation on the LEFT is in $\{s, ac\downarrow\}$ (slightly different formulation may use also $\{aw\downarrow\}$) – dual case on the RIGHT;
- The derivation in the CENTRE is in $\{aw\downarrow, aw\uparrow\}$

Putting Things Together

Various technical steps.. eventually the resulting cut-free form of Π is in $SKS \setminus \{ai\uparrow\}$

$$
n\cdot\mathsf{c}\downarrow\,\frac{\left[A \vee \left[A \vee \cdots \vee \left[A \vee \begin{array}{c} \theta_0 \\ \Phi_0 \Vert \\ \theta_1 \\ \Phi_1 \Vert \\ \theta_2 \\ \vdots \\ \theta_n \\ \Phi_n \Vert \\ [A \vee \theta_{n+1}] \end{array}\right]\right]\right]}{A}
$$

Conclusions

- Some transformations and theorems (splitting, etc) are proper of the deep inference;
- Constructions in sequent calculus can be recast in deep inference, with the advantage of becoming local, improving in complexity.
- Some theorems (e.g. Herbrand's theorem) benefit from proof systems designed on purpose - witness substitution may be lifted to the level of the formalism.
- Finer granularity of rules generate more non-determinism in proof search: splitting reduces the proof search space.
- A (promising) notion of analyticity combines aspects of design, their implications on fundamental theorems and on proof search, and are general to address various different logics.
- Further results in complexity (not seen in this course): exponential speed up cut free sequent calculus, and re-casting Extension to Frege systems.
- Resource awareness (linearity), locality, modularity, boundedness in rules are all features of the methodology that contributes the flavour of a computation-aware proof theory.

Thanks for your interest and attention :-)

References I

[1] Andrea Aler Tubella & Alessio Guglielmi (2017): *Subatomic Proof Systems: Splittable Systems.*
ACM Transactions on Computational Logic 19(1), pp. 5:1–33, doi:10.1145/3173544.
Available at `https://arxiv.org/pdf/1703.10258.pdf`.

[2] Albert Atserias, Nicola Galesi & Pavel Pudlák (2002): *Monotone Simulations of Non-Monotone Proofs.*
Journal of Computer and System Sciences 65(4), pp. 626–638.

[3] Kai Brünnler (2006): *Cut Elimination Inside a Deep Inference System for Classical Predicate Logic.*
Studia Logica 82(1), pp. 51–71, doi:10.1007/s11225-006-6605-4.
Available at `http://cs.bath.ac.uk/ag/kai/q.pdf`.

[4] Kai Brünnler & Alwen Fernanto Tiu (2001): *A Local System for Classical Logic.*
In R. Nieuwenhuis & Andrei Voronkov, editors: *Logic for Programming, Artificial Intelligence, and Reasoning (LPAR)*, *Lecture Notes in Computer Science* 2250, Springer-Verlag, pp. 347–361, doi:10.1007/3-540-45653-8_24.
Available at `http://cs.bath.ac.uk/ag/kai/lcl-lpar.pdf`.

[5] Paola Bruscoli & Alessio Guglielmi (2009): *On Analyticity in Deep Inference.*
Available at `http://cs.bath.ac.uk/ag/p/ADI.pdf`.

[6] Paola Bruscoli & Alessio Guglielmi (2009): *On the Proof Complexity of Deep Inference.*
ACM Transactions on Computational Logic 10(2), pp. 14:1–34, doi:10.1145/1462179.1462186.
Available at `http://cs.bath.ac.uk/ag/p/PrComplDI.pdf`.

References II

[7] Paola Bruscoli, Alessio Guglielmi, Tom Gundersen & Michel Parigot (2010): *A Quasipolynomial Cut-Elimination Procedure in Deep Inference Via Atomic Flows and Threshold Formulae.*
In Edmund M. Clarke & Andrei Voronkov, editors: *Logic for Programming, Artificial Intelligence, and Reasoning (LPAR-16)*, *Lecture Notes in Computer Science* 6355, Springer-Verlag, pp. 136–153, doi:10.1007/978-3-642-17511-4_9.
Available at `http://cs.bath.ac.uk/ag/p/QPNDI.pdf`.

[8] Paola Bruscoli, Alessio Guglielmi, Tom Gundersen & Michel Parigot (2016): *Quasipolynomial Normalisation in Deep Inference Via Atomic Flows and Threshold Formulae.*
Logical Methods in Computer Science 12(1), pp. 5:1–30, doi:10.2168/LMCS-12(2:5)2016.
Available at `https://lmcs.episciences.org/1637/pdf`.

[9] Samuel R. Buss (1995): *On Herbrand's Theorem.*
In Daniel Leivant, editor: *Logic and Computational Complexity*, *Lecture Notes in Computer Science* 960, Springer-Verlag, pp. 195–209, doi:10.1007/3-540-60178-3_85.

[10] Anupam Das (2011): *On the Proof Complexity of Cut-Free Bounded Deep Inference.*
In Kai Brünnler & George Metcalfe, editors: *Tableaux 2011*, *Lecture Notes in Artificial Intelligence* 6793, Springer-Verlag, pp. 134–148, doi:10.1007/978-3-642-22119-4_12.
Available at `http://www.anupamdas.com/items/PrCompII/ProofComplexityBoundedDI.pdf`.

References III

[11] Anupam Das (2014): *On the Pigeonhole and Related Principles in Deep Inference and Monotone Systems.*
In Thomas Henzinger & Dale Miller, editors: *Joint Meeting of the 23rd EACSL Annual Conference on Computer Science Logic (CSL) and the 29th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, ACM, pp. 36:1–10, doi:10.1145/2603088.2603164.
Available at http://www.anupamdas.com/items/WeakMonProofsPHP/WeakMonProofsPHP.pdf.

[12] Alessio Guglielmi (2007): *A System of Interaction and Structure.*
ACM Transactions on Computational Logic 8(1), pp. 1:1–64, doi:10.1145/1182613.1182614.
Available at http://cs.bath.ac.uk/ag/p/SystIntStr.pdf.

[13] Alessio Guglielmi, Tom Gundersen & Michel Parigot (2010): *A Proof Calculus Which Reduces Syntactic Bureaucracy.*
In Christopher Lynch, editor: *21st International Conference on Rewriting Techniques and Applications (RTA)*, *Leibniz International Proceedings in Informatics (LIPIcs)* 6, Schloss Dagstuhl–Leibniz-Zentrum für Informatik, pp. 135–150, doi:10.4230/LIPIcs.RTA.2010.135.
Available at http://drops.dagstuhl.de/opus/volltexte/2010/2649.

[14] Alessio Guglielmi & Lutz Straßburger (2011): *A System of Interaction and Structure V: The Exponentials and Splitting.*
Mathematical Structures in Computer Science 21(3), pp. 563–584, doi:10.1017/S096012951100003X.
Available at http://www.lix.polytechnique.fr/~lutz/papers/NEL-splitting.pdf.

References IV

[15] Tom Gundersen (2009): *A General View of Normalisation Through Atomic Flows.*
Ph.D. thesis, University of Bath.
Available at
`http://tel.archives-ouvertes.fr/docs/00/50/92/41/PDF/thesis.pdf`.

[16] Emil Jeřábek (2009): *Proof Complexity of the Cut-Free Calculus of Structures.*
Journal of Logic and Computation 19(2), pp. 323–339, doi:10.1093/logcom/exn054.
Available at `http://www.math.cas.cz/~jerabek/papers/cos.pdf`.

[17] Benjamin Ralph (2018): *A Natural Proof System for Herbrand's Theorem.*
In Sergei Artemov & Anil Nerode, editors: *Logical Foundations of Computer Science (LFCS)*, *Lecture Notes in Computer Science* 10703, Springer International Publishing, pp. 289–308, doi:10.1007/978-3-319-72056-2_18.
Available at `http://cs.bath.ac.uk/ag/br/prfsysht.pdf`.

[18] Benjamin Ralph (2019): *Modular Normalisation of Classical Proofs.*
Ph.D. thesis, University of Bath.
Available at `http://cs.bath.ac.uk/ag/br/phd.pdf`.

[19] Lutz Straßburger (2003): *MELL in the Calculus of Structures.*
Theoretical Computer Science 309, pp. 213–285.
Available at `http://www.lix.polytechnique.fr/~lutz/papers/els.pdf`.

[20] Ingo Wegener (1987): *The Complexity of Boolean Functions.*
John Wiley & Sons Ltd and B. G. Teubner, Stuttgart.