Three Lectures on Structural Proof Theory

2 – Classical Logic in Deep Inference

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Course Notes
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Outline for Today

Some Observations to Motivate Deep Inference
The Calculus of Structures (Deep Inference Formalism)
Correspondence with the Sequent Calculus
General and Atomic rules (Locality) – Propositional Case
Decomposition and Normal Forms
Design: Extending to First Order
On Cut Elimination
Observation 1 - a Mismatch?

We have seen sequents $\Gamma \vdash \Delta$:

- $\Gamma / \Delta$ 'understood' as some kind of conjunction/disjunction;
- **main connective** of formula drives the bottom-up proof construction.
- **Which is the logical relation between premisses (subproofs) in a branching logical rule?** In $LK$ (in all others too)

\[
\begin{align*}
\land_{LL} & : A, \Gamma \vdash \Delta \quad \land_{LR} & : B, \Gamma \vdash \Delta \\
\quad & : A \land B, \Gamma \vdash \Delta
\
\lor_{L} & : A, \Gamma \vdash \Delta \quad \lor_{R} & : B, \Gamma \vdash \Delta \\
\quad & : A \lor B, \Gamma \vdash \Delta
\end{align*}
\]

- $\lor_L$ and $\land_R$: 'conjunction' of **both subproofs (from the two premisses)**.
- So far, always with left rules, but it **escalates** with more expressive logics (linear logic) in 2-sided sequent calculus.
Observation 2 - Locality

- Recall GS1 \( p \), negation normal form;

\[
\begin{align*}
\text{Ax} & \quad \frac{}{\vdash A, A} \\
\text{Cut} & \quad \frac{\vdash \phi, A \quad \vdash \psi, \neg A}{\vdash \phi, \psi} \\
\text{RL} & \quad \frac{\vdash \phi, A \quad \vdash A \vee B}{\vdash \phi, A \vee B} \\
\text{RR} & \quad \frac{\vdash \phi, B \quad \vdash A \vee B}{\vdash \phi, A \vee B} \\
\text{R\&} & \quad \frac{\vdash \phi, A \quad \vdash \phi, B}{\vdash \phi, A \wedge B} \\
\text{RC} & \quad \frac{\vdash \phi, A \quad \vdash A}{\vdash \phi, A} \\
\text{RW} & \quad \frac{\vdash \phi}{\vdash \phi, A}
\end{align*}
\]

- Local vs non-bounded rules, e.g. \( RC \), when \( A \) is a generic formula: non-suitable for distributed computation where information on \( A \) may be sparse

- ‘problematic rules’ should be atomic, starting from the axiom – is it possible, while keeping the proof theory? (not as much as expected, in sequent calculus presentations)
Starting point

Consider this Variant of $GS1\ p$:

- $\land_R$ with multiplicative context (rather than additive, also for Cut);
- invertible $\lor_R$ (only one rule instead of two);
- constants $\top, \bot$ in the language (and introduces a new axiom).
Deep Inference – The Calculus of Structures

Calculus of Structures – the first formalism developed in Deep Inference
(and for a logic related to process algebras [3, 2, 6])

▶ No main connective;
▶ rules applied ’deep’ inside formulae (possible because implication is
preserved under contextual closure by conjunction/disjunction);
▶ no branching rules (i.e. ‘branches may be re-united’ differently from
sequent calculi)
▶ a careful design of proof systems within the calculus of structures,
for a given logic, delivers a meaningful proof theory, with new
methods for manipulation and analysis of proofs

A number of logics, including linear and modal ones, have been covered
in this formalism. Please refer to the web site for more details - they fall
out of the scope of this short course.

1Deep inference web site: http://alessio.guglielmi.name/res/cos/
**Systems \( KSq \) and \( SKS \) in \( CoS \)**

- **Structures (Formulae), in context notation ([…] is disjunction, (…) is conjunction):**

\[
S ::= f \mid t \mid a \mid [S, \ldots, S] \mid (S, \ldots, S) \mid \bar{S}
\]

- **Syntactic equivalence on formulae:**

<table>
<thead>
<tr>
<th>Associativity</th>
<th>Commutativity</th>
</tr>
</thead>
<tbody>
<tr>
<td>([\bar{R}, [\bar{T}], \bar{U}]] = [\bar{R}, \bar{T}, \bar{U}])</td>
<td>([R, T] = [T, R])</td>
</tr>
<tr>
<td>((\bar{R}, (\bar{T}), \bar{U})) = (\bar{R}, \bar{T}, \bar{U}))</td>
<td>((R, T) = (T, R))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Units</th>
<th>Negation</th>
</tr>
</thead>
<tbody>
<tr>
<td>((f, f) = f)</td>
<td>(\bar{f} = t)</td>
</tr>
<tr>
<td>([t, t] = t)</td>
<td>(\bar{t} = f)</td>
</tr>
<tr>
<td>([t, R] = R)</td>
<td>([R, T] = (\bar{R}, \bar{T}))</td>
</tr>
<tr>
<td>((t, R) = R)</td>
<td>((R, T) = [\bar{R}, \bar{T}])</td>
</tr>
</tbody>
</table>

**Context Closure**

- If \( R = T \) then \( S\{R\} = S\{T\} \)
- \( \bar{R} = \bar{T} \)
- \( \bar{R} = R \)
General Terminology

▶ Inference Rule \( \rho \) (premiss \( V \), conclusion \( U \)) and instance of a deep inference rule \( \pi \), applied within a context \( S\{ \} \):

\[
\begin{align*}
\rho & \quad V \\
\hline
U \quad \pi & \quad S\{T\} \\
& \quad S\{R\}
\end{align*}
\]

▶ Reading: a rewrite rule, where \( R \) is redex and \( T \) is contractum, and an implication \( T \implies R \) (where \( \implies \) is a logical implication, for example, classical implication in a proof system for classical logic).

▶ A proof system is a (finite) set of inference rules.
Derivations and Proofs

**Derivation** – finite sequence of instances of inferences rules in the proof system.

**Derivation within context** – all inference steps happen in some context.

**Proof** – a derivation from premiss t.

\[
\begin{align*}
\Delta &= \pi \quad T \quad V \\
\pi' \quad \Delta \quad &\vdash \\
\rho' \quad U \quad &\vdash \\
\rho \quad R \quad &\vdash \\
S\{\Delta\} &= \pi' \quad S\{T\} \quad \vdash \\
\pi' \quad S\{V\} \quad &\vdash \\
\rho' \quad S\{U\} \quad &\vdash \\
\rho \quad S\{R\} \quad &\vdash \\
\Pi \quad \models \Sigma
\end{align*}
\]

**Inference rule** \( \rho \) **derivable in a system** \( S \) – if there exists a derivation, in the system, from its premiss to conclusion, for all possible instances of the rule.

**Inference rule** \( \rho \) **admissible in a system** \( S \) – for all proofs of \( R \) in \( S \cup \{\rho\} \) there exists a proof of \( R \) in \( S \) (i.e. the provability doesn’t change).
SKSg - General (non-atomic) rules

SKS - (S)ymmetric (K)lassic (S)ystem

\[
\begin{align*}
&i \downarrow \frac{S\{t\}}{S[R, \bar{R}]} & i \uparrow & \frac{S(R, \bar{R})}{S\{f\}} \\
&s & \frac{S([R, U], T)}{S[(R, T), U]} \\
&w \downarrow & \frac{S\{f\}}{S\{R\}} & w \uparrow & \frac{S\{R\}}{S\{t\}} \\
&c \downarrow & \frac{S[R, R]}{S\{R\}} & c \uparrow & \frac{S\{R\}}{S(R, \bar{R})}
\end{align*}
\]
**SKSg - General rules**

Up-rules are admissible.

\[
\frac{S\{t\}}{S[R, \bar{R}]} \quad \frac{S(R, \bar{R})}{S\{f\}}
\]

\[
\frac{S([R, U], T)}{S[(R, T), U]}
\]

\[
\frac{S\{f\}}{S\{R\}} \quad \frac{S\{R\}}{S\{t\}}
\]

\[
\frac{S[R, R]}{S\{R\}} \quad \frac{S\{R\}}{S(R, R)} = \frac{T}{R}
\]

(modulo syntactic equality, represented by \(=\));

Duality up-/down- rules (contrapositive): \(T \longrightarrow R\) and \(\bar{R} \longrightarrow \bar{T}\)

Symmetric KS: for each rule, there is also its dual (implicationally complete)
Some Examples

\[ i \downarrow \frac{S\{t\}}{S[R, \bar{R}]} \quad \quad i \uparrow \frac{S(R, \bar{R})}{S\{f\}} \]

\[ w \downarrow \frac{S\{f\}}{S\{R\}} \quad \quad w \uparrow \frac{S\{R\}}{S\{t\}} \]

\[ c \downarrow \frac{S[R, R]}{S\{R\}} \quad \quad c \uparrow \frac{S\{R\}}{S(R, R)} \]

\[ \frac{i \downarrow \frac{S[R, \bar{R}]}{S\{f\}}}{[(a a) \bar{a} \bar{a}]} \quad \frac{i \uparrow \frac{S(R, \bar{R})}{S\{f\}}}{[(a a) \bar{a}]} \]

\[ \frac{c \downarrow \frac{S[R, R]}{S\{R\}}}{[(b [a \bar{a}])]} \quad \frac{w \downarrow \frac{S\{f\}}{S\{R\}}}{[(b a) \bar{a}]} \quad \frac{w \uparrow \frac{S\{R\}}{S\{t\}}}{[(b a) \bar{a}]} \quad \frac{s \downarrow \frac{S[R, \bar{R}]}{S\{f\}}}{[(a b) a) \bar{a}]} \quad \frac{s \uparrow \frac{S\{R\}}{S(R, R)}}{[(a a) \bar{a} \bar{b}]} \]
Some Examples

\[
\begin{align*}
i \downarrow & \quad \frac{S\{t\}}{S[R, \bar{R}]} \\
& \quad \frac{S(R, \bar{R})}{S\{f\}} \\
\quad s \quad & \quad \frac{S([R, U], T)}{S[(R, T), U]} \\
w \downarrow & \quad \frac{S\{f\}}{S\{R\}} \\
& \quad \frac{S\{R\}}{S\{t\}} \\
c \downarrow & \quad \frac{S[R, R]}{S\{R\}} \\
& \quad \frac{S\{R\}}{S(R, R)}
\end{align*}
\]

Rule AX is derivable and also admissible in KS:

\[
\begin{align*}
t & \quad \frac{\bar{R} S R}{AX} \\
\quad \frac{\bar{R} \bar{R}}{t} \\
\quad \frac{\bar{R} S R}{w} \\
\quad \frac{\bar{R} \bar{R}}{w} \\
\quad \frac{\bar{R} S R}{t}
\end{align*}
\]
Deduction Theorem for $SKS_g$

There is a derivation $\vdash_{SKS_g} T$ if and only if there is a proof $\vdash_{SKS_g} [\bar{T}, R]$.

Sketch of proof

\[ \Delta \vdash_{SKS_g} T \]
\[ \sim \]
\[ i \downarrow \frac{t}{[\bar{T}, T]} \]
\[ [\bar{T}, \Delta] \vdash_{SKS_g} [\bar{T}, R] \]

\[ \Pi \vdash_{SKS_g} T \]
\[ \sim \]
\[ s \downarrow \frac{(T, [\bar{T}, R])}{[R, (T, \bar{T})]} \]
\[ \vdash_{SKS_g} \]
A comparison with sequent rules

Interaction can be applied anywhere, not just the top of derivation

\[ \frac{}{\vdash A, \bar{A}} \quad \text{corresponds to} \quad \frac{t}{i \downarrow [A, \bar{A}]} \]

\[ \frac{\vdash \Phi, A}{\vdash \Phi, \Psi} \quad \text{corresponds to} \quad \frac{([\Phi, A], [\Psi, \bar{A}])}{s \downarrow [\Phi, (A, [\Psi, \bar{A}])]} \]

\[ \frac{\vdash \Phi, A, A}{\vdash \Phi, A} \quad \text{corresponds to} \quad \frac{[\Phi, A, A]}{c \downarrow [\Phi, A]} \]

\[ \frac{\vdash \Phi}{\vdash \Phi, A} \quad \text{corresponds to} \quad \frac{\Phi}{w \downarrow [\Phi, f]} \]

\( w \uparrow \) and \( c \uparrow \): nothing similar in GS1p
Dual derivation

**Dual of a derivation** – flip the derivation upside down it, replacing each rule/connective/atom by its dual

\[
\begin{align*}
&\text{w} \uparrow \frac{[(a, b), a]}{[a, a]} \\
&\text{c} \downarrow \frac{[a, a]}{a}
\end{align*}
\]

is dual to

\[
\begin{align*}
&\text{c} \uparrow \frac{\bar{a}}{[\bar{a}, \bar{a}]} \\
&\text{w} \downarrow \frac{([\bar{a}, b], \bar{a})}{([\bar{a}, b], \bar{a})}
\end{align*}
\]

The **dual of a proof** will not be a proof, rather a **refutation**.
From Sequent Calculus to CoS

Theorem 2.3.3. For every derivation \( \Sigma_1 \ldots \Sigma_h \in GS1p + Cut \) there exists a derivation \( (\Sigma_{1s}, \ldots, \Sigma_{hs}) \) with the same number of cuts.

Sketch of proof

- Translate formulae/sequents into structures of SKS;
- Structural induction on derivation \( \Delta \);
- e.g. the last rule is \( R\wedge \) (similarly, for cut)

\[
\begin{align*}
\Delta &= \begin{array}{c}
\Sigma_1 \cdots \Sigma_k, \Sigma'_1 \cdots \Sigma'_l
\end{array} \\
\Sigma \vdash \Phi, A, \Sigma'_1 \cdots \Sigma'_l
\end{align*}
\]

\[
\begin{align*}
\Delta' &= \begin{array}{c}
[\Phi, A], \Sigma'_1 \cdots \Sigma'_l
\end{array} \\
\Sigma\vdash [\Phi, A], [\Psi, B]
\end{align*}
\]

\[
\begin{align*}
\Delta_2' &= \begin{array}{c}
[\Phi, A], [\Psi, B]
\end{array} \\
\Sigma\vdash [\Phi, \Psi, (A, B)]
\end{align*}
\]

corresponds to
... and since proofs are derivations of a specific form, these hold:

- Proofs (with cut) correspond to proofs in $SKS_g \setminus \{c \uparrow, w \uparrow\}$
- Cut-free proofs correspond to proofs in $SKS_g \setminus \{i \uparrow, c \uparrow, w \uparrow\}$
From CoS to Sequent Calculus

Theorem 2.3.8. For every derivation $\vdash_{\text{SKS}_g} Q$ there exists a derivation $\vdash_{\text{GS1p} + \text{Cut}} P$.

(And similarly, for proofs)

Sketch of proof

- Translate structures of SKS into formulae;
- Mimic deep inference within context in the sequent calculus;
- Proceed top-down, starting from the top-most rule, by induction on the length of the SKS derivation.
From CoS to Sequent Calculus – cont’d

SKS derivation $\Delta$ (left) and corresponding construction (right)

$$Q \vdash_{SKS_g} \frac{S\{T\}}{S\{R\}} \rho \frac{S\{R\}}{S\{T\}}$$

$$\Delta \vdash_{SKS_g} \Delta'$$

$\vdash R, \overline{T}$

$\vdash S\{R\}, S\{T\}$

$\vdash S\{T\}$

Cut

$\vdash S\{R\}$

$\vdash P$
From CoS to Sequent Calculus – cont’d

SKS derivation $\Delta$ (left) and corresponding construction (right)

- Proof $\Pi$ exists, specific to rule $\rho$ ($T \rightarrow R$)
From CoS to Sequent Calculus – cont’d

SKS derivation $\Delta$ (left) and corresponding construction (right)

- Mimic the specific instance of $\rho$, in context $S\{\}$ (lemma)
From CoS to Sequent Calculus – cont’d

KS derivation $\Delta$ (left) and corresponding construction (right)

- Inductive hypothesis
In particular, the up-rules will have these proofs associated:

\[
\begin{align*}
\text{i} & \uparrow \frac{S(U, \bar{U})}{S\{f\}}, \\
\text{c} & \uparrow \frac{S\{U\}}{S(U, U)}, \\
\text{w} & \uparrow \frac{S\{U\}}{S\{t\}}.
\end{align*}
\]

- **KSg** is **SKSg** without up-rules;
- Up-rules are **admissible** for **KSg**
  - A proof in **SKSg** is translated into one in **GS\(p1 + Cut\)**,
  - Cut-elimination holds: get a cut free proof in **GS\(p1\)**,
  - Translate back, it is a proof in **KSg**
From CoS to Sequent Calculus – cont’d

In particular, the up-rules will have these proofs associated:

\[
\text{i} \vdash \frac{S(U, \bar{U})}{S\{f\}} \quad , \quad \text{c} \vdash \frac{S\{U\}}{S(U, U)} \quad , \\
\text{w} \vdash \frac{S\{U\}}{S\{t\}} \quad ,
\]

Two proof systems $S_1$ and $S_2$ are said

- (weakly) equivalent – for every proof of $R$ in $S_1$ there is a proof of $R$ in $S_2$, and vice versa;
- strongly equivalent – for every derivation from $T$ to $R$ in $S_1$, there is a derivation from $T$ to $R$ in $S_2$, and vice versa;
- e.g. SKSg and KSG are equivalent but not strongly equivalent
KSg – Remarks

- Cut-free sequent system: all rules fulfill subformula property;
- Down-fragment in deep inference: premisses of rules do not have new atoms that are not in the conclusion
KSg – Remarks

Notion of invertible rule of sequent calculus is imported:

Definition 2.4.7. A rule \( \rho \) is invertible for a system \( \mathcal{S} \) if for each instance
\[
\frac{V}{U}
\]
there is a derivation \( \mathcal{S} \).

.. and it is used to separate parts of the system (\( S' \) is the cnf of \( S \))

For every proof \( \vdash_{SKSg} \) there is a proof \( \vdash \{s,c\} \).
Locality via Atomic Rules - SKS

**Objective** – make interaction/weakening/contraction all atomic

In sequent calculus:

- **atomic axiom** may replace a general one, but **making atomic a cut**, given a general system, is not a free lunch.
- **GS1p** (with multiplicative $\land_R$) **does not allow contraction to be made atomic**:

$$\vdash (a \land b), (\bar{a} \lor \bar{b}) \land (\bar{a} \lor \bar{b})$$

In deep inference:

- Making interaction/cut and weakening (and dual) atomic is easy
- **Making atomic contraction requires the medial rule** (derivable in sequent calculus, but not as a rule):
Locality via Atomic Rules - *SKS*

\[
\begin{align*}
\text{ai} & \downarrow \frac{S\{t\}}{S[a, \bar{a}]} & \text{ai} & \uparrow \frac{S(a, \bar{a})}{S\{f\}} \\
S([R, U], T) & \overset{s}{\frac{S((R, T), U)}{S([R, T], U)}} \\
S([(R, U), (T, V)]) & \overset{m}{\frac{S([R, T], [U, V])}{S([R, T], [U, V])}} \\
\text{aw} & \downarrow \frac{S\{f\}}{S\{a\}} & \text{aw} & \uparrow \frac{S\{a\}}{S\{t\}} \\
\text{ac} & \downarrow \frac{S[a, a]}{S\{a\}} & \text{ac} & \uparrow \frac{S\{a\}}{S(a, a)}
\end{align*}
\]
Locality via Atomic Rules - \textit{SKS} and \textit{KS}
Derivability of General Rules

Theorem 4.1.2. The rules $i\downarrow$, $w\downarrow$ and $c\downarrow$ are derivable for \{ai\downarrow, s\}, \{aw\downarrow, s\} and \{ac\downarrow, m\}, respectively. Dually, the rules $i\uparrow$, $w\uparrow$ and $c\uparrow$ are derivable for \{ai\uparrow, s\}, \{aw\uparrow, s\} and \{ac\uparrow, m\}, respectively.

Sketch of proof: cases for weakening, $R$ not an atom (and dual rule)

\[
\begin{align*}
\text{w} \downarrow & \quad \frac{s\{f\}}{s\{R\}} \\
\text{w} \downarrow & \quad \frac{s\{f\}}{s([t, t], f)} \\
\text{w} \downarrow & \quad \frac{s\{f\}}{s[t, (t, f)]} \\
\text{w} \downarrow & \quad \frac{s\{f\}}{s[f, f]} \\
\text{w} \downarrow & \quad \frac{s\{f\}}{s[f, Q]} \\
\text{w} \downarrow & \quad \frac{s\{f\}}{s[P, Q]} \\
\text{w} \downarrow & \quad \frac{s\{f\}}{s(f, f)} \\
\text{w} \downarrow & \quad \frac{s\{f\}}{s(f, Q)} \\
\text{w} \downarrow & \quad \frac{s\{f\}}{s(P, Q)}
\end{align*}
\]

whereas, for contraction, medial is needed in this case

\[
\begin{align*}
m & \quad \frac{s\left((P, Q), (P, Q)\right)}{s([P, P], (P, Q))} \\
c \downarrow & \quad \frac{s\left([P, [P, Q]]\right)}{s([P, P], [Q, Q])} \\
c \downarrow & \quad \frac{s\left([P, P], Q\right)}{s(P, Q)} \\
c \downarrow & \quad \frac{s\left([P, P], Q\right)}{s(P, Q)}
\end{align*}
\]
Derivability of General Rules

▶ Therefore, $KS$ and $KSG$ are strongly equivalent.
▶ We may occasionally use general rules in $KS$, just as shorthand notation.
▶ (Atomic) contraction is related to sharing
On Design: Extension to first order

- Structures (Formulae) extended with quantifiers:

\[
S ::= f \mid t | a \mid [S, \ldots, S] \mid (S, \ldots, S) \mid \overline{S} \mid \exists xS \mid \forall xS
\]

- Syntactic equivalence on formulae are extended with

  *Variable Renaming*  
  \[\forall xR = \forall yR[x/y]\] \text{ if } y \text{ is not free in } R

  \[\exists xR = \exists yR[x/y]\]

  *Vacuous Quantifier*  
  \[\forall yR = \exists yR = R\] \text{ if } y \text{ is not free in } R

  *Negation*  
  \[\overline{\exists xR} = \forall x\overline{R}\]

  \[\overline{\forall xR} = \exists x\overline{R}\]

- Remark – the quantifier rules in GS1 are

\[
R\exists \vdash \Phi, A[x/\tau] \\
\vdash \Phi, \exists xA
\]

\[
R\forall \vdash \Phi, A[x/y] \\
\vdash \Phi, \forall xA
\]

Proviso: \( y \) is not free in the conclusion of \( R\forall \).
SKSūq – General, first order

\[ i \downarrow \frac{S\{t\}}{S[R, \bar{R}]} \quad i \uparrow \frac{S(R, \bar{R})}{S\{f\}} \]

\[ s \frac{S((R, U), T)}{S[(R, T), U]} \]

\[ u \downarrow \frac{S\{\forall x[R, T]\}}{S[\forall xR, \exists xT]} \quad u \uparrow \frac{S(\exists xR, \forall xT)}{S\{\exists x(R, T)\}} \]

\[ w \downarrow \frac{S\{f\}}{S\{R\}} \quad w \uparrow \frac{S\{R\}}{S\{t\}} \]

\[ c \downarrow \frac{S[R, R]}{S\{R\}} \quad c \uparrow \frac{S\{R\}}{S(R, R)} \]

\[ n \downarrow \frac{S[R[x/\tau]]}{S[\exists xR]} \quad n \uparrow \frac{S[\forall xR]}{S[R[x/\tau]]} \]
SKSgq – General, first order

\[
\begin{align*}
i \downarrow & \frac{S\{t\}}{S[R, \bar{R}]} & i \uparrow & \frac{S(R, \bar{R})}{S\{f\}} \\
& \frac{S([R, U], T)}{S[(R, T), U]} & & \frac{S(\exists x R, \forall x T)}{S\{\exists x (R, T)\}} \\
u \downarrow & \frac{S\{\forall x [R, T]\}}{S[\forall x R, \exists x T]} & u \uparrow & \frac{S\{\exists x R, \forall x T\}}{S\{\exists x (R, T)\}} \\
w \downarrow & \frac{S\{f\}}{S\{R\}} & w \uparrow & \frac{S\{R\}}{S\{t\}} \\
c \downarrow & \frac{S[R, R]}{S\{R\}} & c \uparrow & \frac{S\{R\}}{S(R, R)} \\
n \downarrow & \frac{S[R[x/\tau]]}{S\{\exists x R\}} & n \uparrow & \frac{S[\forall x R]}{S\{R[x/\tau]\}}
\end{align*}
\]
**SKSgq – General, first order**

- **u↓** – premiss implies conclusion (differently from GS1).
  
  \[
  \begin{align*}
  u \downarrow & \quad \frac{S\{\forall x[R,T]\}}{S[\forall x R, \exists x T]} \\
  & = \frac{S[\forall x R]}{S[\forall x R, T]} \quad \text{if } x \text{ is not free in } T,
  \end{align*}
  \]

- **n↓** – the term \( \tau \) is not required to be free for \( x \) in \( S\{R\} \), i.e. there can be quantifiers in context \( S\{\} \) that may capture variables in \( \tau \).

- Both rules seem more **local**

- Results of the propositional case are extended to the predicate case. In particular **Deduction theorem, admissibility of up-fragment** (and hence **indirect proof of cut-elimination**).
(S)KSq – Atomic, first order

Instances of contraction over quantified formulae – any interference with medial towards atomic contraction?

► Two more rules are needed (and their dual ones in the symmetric system);
► Strong equivalence on the first order KSq and KSgq
Reminder: Cut Elimination in Sequent Calculus

- Above the Cut (in a branching situation), two ’dual’ logical rules operate on the cut formula (and its dual),
- just on their respective main connective.
- Restricting the cut rule to be atomic would help.
- This method cannot be adapted to deep inference so easily.
Cut Elimination in \textit{SKS} - Idea (Propositional)

- This cut-elimination procedure is inspirational: a mixture between natural deduction and proper context rewriting.
- Based mostly on works by Brünnler and Tiu, e.g. see [1].
- But it does not easily scale up to first order case.
Cut Elimination in $SKS$ - Idea
Cut Elimination in SKS - Idea

1- duplicate the proof above cut; remove a literal in each
Cut Elimination in SKS - Idea

2- replace R in all a; it breaks the proof, in the ai rules
Cut Elimination in SKS - Idea

3- fix the broken instances by using copies of the other proof
Cut Elimination in SKS - Idea

4- Use a contraction to conclude
Lemmata for the Proof of Cut Elim’on in SKS

1–Transform the original SKS into one in KS ∪ i↑ (or atomic)

Atomic cut is derivable for shallow atomic cut (below, left) and s:

2–Deal only with shallow atomic cut

Any proof of T{a} in KS can be transformed into one of T{t} in KS. Trace-replace the occurrences of a, bottom-up in a proof – e.g.

3–Generate this way the two initial copies of proofs above the cut
Cut Elimination in \textit{SKS}

Start with a transformed proof: only shallow atomic cuts as up-rule. Consider the topmost cut, and generate the two copies of the proof above the cut (use $a/t$ and $\bar{a}/t$):

\[
\begin{align*}
\Pi & \vdash_{KS} \\
ai \uparrow & \frac{[R, (a, \bar{a})]}{R} \\
\Delta & \vdash_{KS \cup \{ai \uparrow\}} \\
T & \vdash_{KS} \\
\Pi_1 & \vdash_{KS} \\
& \frac{[R, a]}{[R, \bar{a}]} \quad \text{and} \\
\Pi_2 & \vdash_{KS} \\
\end{align*}
\]

Bottom-up in $\Pi_1$ replace $a/R$ – no effect if $a$ is in the context or in $s, m$. Otherwise, fix it (left) to paste it for the cut-eliminated proof (right),
Cut Elimination in *SKS*

Start with a transformed proof: only shallow atomic cuts as up-rule. Consider the topmost cut, and generate the two copies of the proof above the cut (use \( a/t \) and \( \bar{a}/t \)):

\[
\begin{align*}
\Pi &\vdash_{KS} \\
\frac{[R, (a, \bar{a})]}{R} &\quad \text{ai}^\uparrow \\
\Delta &\vdash_{KS\cup\{\text{ai}^\uparrow\}} T \\
\end{align*}
\]

and

\[
\begin{align*}
\Pi_1 &\vdash_{KS} [R, a] \\
\Pi_2 &\vdash_{KS} [R, \bar{a}] \\
\end{align*}
\]

Bottom-up in \( \Pi_1 \) replace \( a/R \) – no effect if \( a \) is in the context or in \( s, m \). Otherwise, fix it (left) to paste it for the cut-eliminated proof (right),

\[
\begin{align*}
\text{ac} \downarrow \frac{S[a, a]}{S\{a\}} &\sim \text{c} \downarrow \frac{S[R, R]}{S\{R\}} \\
\text{aw} \downarrow \frac{S\{f\}}{S\{a\}} &\sim \text{w} \downarrow \frac{S\{f\}}{S\{R\}} \\
\text{ai} \downarrow \frac{S\{t\}}{S[a, \bar{a}]} &\sim \frac{S\{\Pi_2\}}{KS} \quad \frac{S\{t\}}{S[R, \bar{a}]} \\
\end{align*}
\]
Decomposition and Normal Forms

Studying the **permutability** of rules in KS/SKS allows the discovery of ways to decompose proofs/derivations.

**Definition 7.1.1.** A rule $\rho$ permutes over a rule $\pi$ (or $\pi$ permutes under $\rho$)

\[
\begin{array}{c}
\frac{T}{\pi \frac{U}{R}} \\
\frac{\rho \frac{V}{R}}{T}
\end{array}
\]

if for every derivation $\pi \frac{U}{R}$ there is a derivation $\rho \frac{V}{R}$ for some formula $V$.

(... various transformations, for different logics, but with some resemblances)
Decomposition and Normal Forms

Example 1: separating cut and interaction

For every derivation $T \vdash_{SKS} \{a_i \uparrow\}$ there is a derivation $V \vdash_{SKS \setminus \{a_i \downarrow, a_i \uparrow\}} U \vdash_{\{a_i \uparrow\}} R$.

Example 2: separate contraction (not possible in sequent calculus).

For every proof $S \vdash_{KS} \{a \downarrow\}$ there is a proof $S' \vdash_{\{a \downarrow\}} S$.

Example 3: separate weakening in a proof.

For every proof $S \vdash_{KS} \{a \downarrow\}$ there is a proof $S' \vdash_{\{a \downarrow\}} S$.
Some Remarks

- Some of these decompositions entail elimination of cuts;
- They can be used for an interpolation theorem;
- The 'layering' of rules application (decomposition) is informative and may be used to guide the proof-search process;
- It can also support incremental design of extensions of the system, with new connectives;
- Choices in the design of inference rules may impact on other theorems, for example Herbrand’s theorem (a good overview is in Ralph’s PhD thesis [5])

(.. just to mention a few..)
Some Proposed Activities

▶ Is it possible to build a derivation with premiss \( c \) and conclusion \([(a a) \ (b c) \ \bar{a} \ (\bar{b} \ c)]\) in \( SKS \)? And in \( KS \cup \{i \uparrow\} \)?

▶ Are \( KS \cup \{i \uparrow\} \) and \( KS \cup \{c \uparrow\} \) strongly equivalent? Are they equivalent? Are they equivalent to \( SKS \)?

▶ You might try and prove some of the case analyses that establish the correspondence between derivations in \( GS1p + Cut \) and in \( SKS \).

▶ In the translation from CoS to sequent calculus, we have mentioned (but not even sketched) the need of a lemma to mimic the deep application of an inference rule \( \rho \) in a context \( S\{\} \). You might like to reconstruct that proof.

▶ Complete the proof that \( c\uparrow \) is derivable in \( \{ac\uparrow, m\} \).

Deep inference web site: http://alessio.guglielmi.name/res/cos/
References


