Three Lectures on Structural Proof Theory

2 – Classical Logic in Deep Inference

Paola Bruscoli

University of Bath

Course Notes

The Proof Society Summer School, Swansea, September 2019

Outline for Today

Some Observations to Motivate Deep Inference
The Calculus of Structures (Deep Inference Formalism)
Correspondence with the Sequent Calculus
General and Atomic rules (Locality) – Propositional Case
Decomposition and Normal Forms
Design: Extending to First Order
On Cut Elimination

Observation 1 - a Mismatch?

We have seen sequents $\Gamma \vdash \Delta$:

- Γ/Δ 'understood' as some kind of conjunction/disjunction;
- main connective of formula drives the bottom-up proof construction.
- Which is the logical relation between premisses (subproofs) in a branching logical rule? In *LK* (in all others too)

$$\wedge_{LL}\ \frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \wedge_{LR}\ \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \wedge_{R}\ \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee_{L}\ \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad \vee_{RL}\ \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \qquad \vee_{RR}\ \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}$$

- $\vee_L$ and $\wedge_R$: 'conjunction' of both subproofs (from the two premisses).
- So far, always with left rules, but it escalates with more expressive logics (linear logic) in 2-sided sequent calculus.

Observation 2 - Locality

- Recall *GS1p*, negation normal form;

$$\text{Ax}\ \frac{}{\vdash A, \bar{A}} \qquad \text{Cut}\ \frac{\vdash \Phi, A \quad \vdash \Psi, \bar{A}}{\vdash \Phi, \Psi}$$

$$\text{R}\lor_{\text{L}}\ \frac{\vdash \Phi, A}{\vdash \Phi, A \lor B} \qquad \text{R}\lor_{\text{R}}\ \frac{\vdash \Phi, B}{\vdash \Phi, A \lor B} \qquad \text{R}\land\ \frac{\vdash \Phi, A \quad \vdash \Phi, B}{\vdash \Phi, A \land B}$$

$$\text{RC}\ \frac{\vdash \Phi, A, A}{\vdash \Phi, A} \qquad \text{RW}\ \frac{\vdash \Phi}{\vdash \Phi, A}$$

- Local vs non-bounded rules, e.g. *RC*, when *A* is a generic formula: non-suitable for distributed computation where information on *A* may be sparse
- 'problematic rules' should be atomic, starting from the axiom – is it possible, while keeping the proof theory? (not as much as expected, in sequent calculus presentations)

Starting point

Consider this Variant of *GS1p*:

- $\wedge_R$ with multiplicative context (rather than additive, also for Cut));
- invertible $\vee_R$ (only one rule instead of two);
- constants $\top$, $\bot$ in the language (and introduces a new axiom).

$$\top\ \frac{}{\vdash \top} \qquad \mathsf{Ax}\ \frac{}{\vdash A, \bar{A}}$$

$$\mathsf{R}\wedge\ \frac{\vdash \Phi, A \quad \vdash \Psi, B}{\vdash \Phi, \Psi, A \wedge B} \qquad \mathsf{R}\vee\ \frac{\vdash \Phi, A, B}{\vdash \Phi, A \vee B}$$

$$\mathsf{RC}\ \frac{\vdash \Phi, A, A}{\vdash \Phi, A} \qquad \mathsf{RW}\ \frac{\vdash \Phi}{\vdash \Phi, A} \qquad \mathsf{Cut}\ \frac{\vdash \Phi, A \quad \vdash \Psi, \bar{A}}{\vdash \Phi, \Psi}$$

Deep Inference – The Calculus of Structures

Deep Inference – a methodology in Proof Theory [4][1]
Calculus of Structures – the first formalism developed in Deep Inference (and for a logic related to process algebras [3, 2, 6])

- No main connective;
- rules applied 'deep' inside formulae (possible because implication is preserved under contextual closure by conjunction/disjunction);
- no branching rules (i.e. 'branches may be re-united' differently from sequent calculi)
- a careful design of proof systems within the calculus of structures, for a given logic, delivers a meaningful proof theory, with new methods for manipulation and analysis of proofs

A number of logics, including linear and modal ones, have been covered in this formalism. Please refer to the web site for more details - they fall out of the scope of this short course.

---

[1] *Deep inference web site:* `http://alessio.guglielmi.name/res/cos/`

Systems *KSq* and *SKS* in CoS

- Structures (Formulae), in context notation ([…] is disjunction, (…) is conjunction):

$$S ::= \mathsf{f} \mid \mathsf{t} \mid a \mid [\underbrace{S,\ldots,S}_{>0}] \mid (\underbrace{S,\ldots,S}_{>0}) \mid \bar{S}$$

- Syntactic equivalence on formulae:

Associativity

$$[\vec{R},[\vec{T}],\vec{U}] = [\vec{R},\vec{T},\vec{U}]$$
$$(\vec{R},(\vec{T}),\vec{U}) = (\vec{R},\vec{T},\vec{U})$$

Commutativity

$$[R,T] = [T,R]$$
$$(R,T) = (T,R)$$

Units

$$(\mathsf{f},\mathsf{f}) = \mathsf{f} \qquad [\mathsf{f},R] = R$$
$$[\mathsf{t},\mathsf{t}] = \mathsf{t} \qquad (\mathsf{t},R) = R$$

Negation

$$\bar{\mathsf{f}} = \mathsf{t}$$
$$\bar{\mathsf{t}} = \mathsf{f}$$
$$\overline{[R,T]} = (\bar{R},\bar{T})$$
$$\overline{(R,T)} = [\bar{R},\bar{T}]$$
$$\bar{\bar{R}} = R$$

Context Closure

$$\text{if} \quad R = T \quad \text{then} \quad \begin{array}{c} S\{R\} = S\{T\} \\ \bar{R} = \bar{T} \end{array}$$

General Terminology

- Inference Rule ρ (premiss V, conclusion U) and instance of a deep inference rule π, applied within a context $S\{\ \}$:

$$\rho \frac{V}{U} \qquad \pi \frac{S\{T\}}{S\{R\}}$$

- Reading: a rewrite rule, where R is redex and T is contractum, and an implication $T \Longrightarrow R$ (where $\Longrightarrow$ is a logical implication, fro example, classical implication in a proof system for classical logic).
- A proof system is a (finite) set of inference rules.

Derivations and Proofs

Derivation – finite sequence of instances of inferences rules in the proof system.

Derivation within context – all inference steps happen in some context.

Proof – a derivation from premiss t.

$$\Delta = \begin{array}{rc} & T \\ \pi' & \overline{V} \\ \pi & \overline{} \\ & \vdots \\ \rho' & \overline{} \\ & U \\ \rho & \overline{R} \end{array} \qquad S\{\Delta\} = \begin{array}{rc} & S\{T\} \\ \pi' & \overline{S\{V\}} \\ \pi & \overline{\phantom{S\{V\}}} \\ & \vdots \\ \rho' & \overline{\phantom{S\{U\}}} \\ & S\{U\} \\ \rho & \overline{S\{R\}} \end{array} \qquad \begin{array}{c} \Pi \Big\Vert \mathscr{S} \\ R \end{array}$$

Inference rule ρ derivable in a system *S* – if there exists a *derivation*, in the system, from its premiss to conclusion, for all possible instances of the rule.

Inference rule ρ admissible in a system *S* – for all *proofs* of *R* in $S \cup \{\rho\}$ there exists a proof of *R* in *S* (i.e. the provability doesn't change).

SKSg - General (non-atomic) rules

SKS - (S)ymmetric (K)lassic (S)ystem

$$\mathsf{i}{\downarrow}\ \frac{S\{\mathsf{t}\}}{S[R,\bar{R}]} \qquad \mathsf{i}{\uparrow}\ \frac{S(R,\bar{R})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{w}{\downarrow}\ \frac{S\{\mathsf{f}\}}{S\{R\}} \qquad \mathsf{w}{\uparrow}\ \frac{S\{R\}}{S\{\mathsf{t}\}}$$

$$\mathsf{c}{\downarrow}\ \frac{S[R,R]}{S\{R\}} \qquad \mathsf{c}{\uparrow}\ \frac{S\{R\}}{S(R,R)}$$

SKSg - General rules

Up-rules are admissible.

$$\mathsf{i}\!\downarrow\frac{S\{\mathsf{t}\}}{S[R,\bar{R}]}\qquad\qquad \mathsf{i}\!\uparrow\frac{S(R,\bar{R})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{w}\!\downarrow\frac{S\{\mathsf{f}\}}{S\{R\}}\qquad\qquad \mathsf{w}\!\uparrow\frac{S\{R\}}{S\{\mathsf{t}\}}$$

$$\mathsf{c}\!\downarrow\frac{S[R,R]}{S\{R\}}\qquad\qquad \mathsf{c}\!\uparrow\frac{S\{R\}}{S(R,R)}\qquad\qquad =\frac{T}{R}$$

(modulo syntactic equality, represented by $=$);

Duality up-/down- rules (contrapositive): $T \Longrightarrow R$ and $\bar{R} \Longrightarrow \bar{T}$

Symmetric KS: for each rule, there is also its dual (implicationally complete)

Some Examples

$$\mathsf{i}\!\downarrow \frac{S\{\mathsf{t}\}}{S[R,\bar{R}]} \qquad \mathsf{i}\!\uparrow \frac{S(R,\bar{R})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\, \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{w}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{R\}} \qquad \mathsf{w}\!\uparrow \frac{S\{R\}}{S\{\mathsf{t}\}}$$

$$\mathsf{c}\!\downarrow \frac{S[R,R]}{S\{R\}} \qquad \mathsf{c}\!\uparrow \frac{S\{R\}}{S(R,R)}$$

$$\mathsf{c}\!\downarrow \dfrac{\mathsf{i}\!\downarrow \dfrac{\mathsf{t}}{[(a\ a)\ \bar{a}\ \bar{a}]}}{[(a\ a)\ \bar{a}]}$$

$$\mathsf{s}\, \dfrac{\mathsf{w}\!\downarrow \dfrac{\mathsf{s}\, \dfrac{\mathsf{i}\!\downarrow \dfrac{b}{(b\ [a\ \bar{a}])}}{[(b\ a)\ \bar{a}]}}{[([a\ b]\ a)\ \bar{a}]}}{[(a\ a)\ \bar{a}\ b]}$$

Some Examples

$$\mathsf{i}{\downarrow}\,\frac{S\{\mathsf{t}\}}{S[R,\bar{R}]} \qquad \mathsf{i}{\uparrow}\,\frac{S(R,\bar{R})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{w}{\downarrow}\,\frac{S\{\mathsf{f}\}}{S\{R\}} \qquad \mathsf{w}{\uparrow}\,\frac{S\{R\}}{S\{\mathsf{t}\}}$$

$$\mathsf{c}{\downarrow}\,\frac{S[R,R]}{S\{R\}} \qquad \mathsf{c}{\uparrow}\,\frac{S\{R\}}{S(R,R)}$$

Rule AX is derivable
and also admissible in KS:

$$\mathsf{AX}\,\frac{\mathsf{t}}{[\bar{R}\ \bar{S}\ R]}$$

$$\mathsf{w}{\downarrow}\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{\mathsf{t}}{[\bar{R}\ R]}}{[\bar{R}\ \bar{S}\ R]} \qquad \mathsf{i}{\downarrow}\,\dfrac{\mathsf{w}{\downarrow}\,\dfrac{=\,\dfrac{\mathsf{t}}{[\mathsf{f}\ \mathsf{t}]}}{[\bar{S}\ \mathsf{t}]}}{[\bar{R}\ \bar{S}\ R]}$$

Deduction Theorem for *SKSg*

There is a derivation $\begin{array}{c} T \\ \Vert\,\mathsf{SKSg} \\ R \end{array}$ *if and only if there is a proof* $\begin{array}{c} \overline{\Vert}\,\mathsf{SKSg} \\ [\bar{T}, R] \end{array}$.

Sketch of proof

$$\begin{array}{c} T \\ \Delta \Vert\,\mathsf{SKSg} \\ R \end{array} \quad \leadsto \quad \begin{array}{c} \mathsf{i}{\downarrow}\,\dfrac{\mathsf{t}}{[\bar{T}, T]} \\ [\bar{T},\Delta] \Vert\,\mathsf{SKSg} \\ [\bar{T}, R] \end{array} ,$$

$$\begin{array}{c} \Pi \overline{\Vert}\,\mathsf{SKSg} \\ [\bar{T}, R] \end{array} \quad \leadsto \quad \begin{array}{c} T \\ (T,\Pi) \Vert\,\mathsf{SKSg} \\ \mathsf{s}\,\dfrac{(T, [\bar{T}, R])}{\mathsf{i}{\uparrow}\,\dfrac{[R, (T, \bar{T})]}{R}} \end{array} .$$

A comparison with sequent rules

Interaction can be applied anywhere, not just the top of derivation

$$\frac{}{\vdash A, \bar{A}} \quad \text{corresponds to} \quad \mathsf{i}\!\downarrow \frac{\mathsf{t}}{[A, \bar{A}]}$$

$$\mathsf{Cut}\,\frac{\vdash \Phi, A \quad \vdash \Psi, \bar{A}}{\vdash \Phi, \Psi} \quad \text{corresponds to} \quad \begin{array}{c} \mathsf{s}\,\dfrac{([\Phi, A], [\Psi, \bar{A}])}{\mathsf{s}\,\dfrac{[\Phi, (A, [\Psi, \bar{A}])]}{\mathsf{i}\!\uparrow \dfrac{[\Phi, \Psi, (A, \bar{A})]}{= \dfrac{[\Phi, \Psi, \mathsf{f}]}{[\Phi, \Psi]}}}} \end{array}$$

$$\mathsf{RC}\,\frac{\vdash \Phi, A, A}{\vdash \Phi, A} \quad \text{corresponds to} \quad \mathsf{c}\!\downarrow \frac{[\Phi, A, A]}{[\Phi, A]}$$

$$\mathsf{RW}\,\frac{\vdash \Phi}{\vdash \Phi, A} \quad \text{corresponds to} \quad = \frac{\Phi}{\mathsf{w}\!\downarrow \dfrac{[\Phi, \mathsf{f}]}{[\Phi, A]}}$$

w↑ and c↑: nothing similar in GS1p

Dual derivation

Dual of a derivation – flip the derivation upside down it, replacing each rule/connective/atom by its dual

$$\mathsf{w}\!\uparrow \frac{\mathsf{c}\!\downarrow \dfrac{[a,a]}{a}}{[(a,\bar{b}),a]} \quad \text{is dual to} \quad \mathsf{w}\!\downarrow \frac{\mathsf{c}\!\uparrow \dfrac{\bar{a}}{(\bar{a},\bar{a})}}{([\bar{a},b],\bar{a})}$$

The dual of a proof will not be a proof, rather a refutation.

From Sequent Calculus to CoS

Theorem 2.3.3. *For every derivation* $\begin{array}{c}\Sigma_1 \cdots \Sigma_h\\ \bigtriangledown\\ \Sigma\end{array}$ *in* GS1p + Cut *there exists a derivation* $\begin{array}{c}(\underline{\Sigma_1}_S, \dots, \underline{\Sigma_h}_S)\\ \Big\| \mathsf{SKSg} \setminus \{\mathsf{c}\!\uparrow, \mathsf{w}\!\uparrow\}\\ \underline{\Sigma}_S\end{array}$ *with the same number of cuts.*

Sketch of proof

- Translate formulae/sequents into structures of *SKS*;
- Structural induction on derivation Δ;
- e.g. the last rule is $R\wedge$ (similarly, for cut)

$$\Delta = \mathsf{R}\wedge \frac{\begin{array}{c}\Sigma_1 \cdots \Sigma_k\\ \bigtriangledown\\ \vdash \Phi, A\end{array} \qquad \begin{array}{c}\Sigma'_1 \cdots \Sigma'_l\\ \bigtriangledown\\ \vdash \Psi, B\end{array}}{\vdash \Phi, \Psi, A \wedge B}$$

corresponds to

$$\begin{array}{c}(\Sigma_1, \dots, \Sigma_k, \Sigma'_1, \dots, \Sigma'_l)\\ \Delta'_1 \Big\| \mathsf{SKSg} \setminus \{\mathsf{c}\!\uparrow, \mathsf{w}\!\uparrow\}\\ ([\Phi, A], \Sigma'_1, \dots, \Sigma'_l)\\ \Delta'_2 \Big\| \mathsf{SKSg} \setminus \{\mathsf{c}\!\uparrow, \mathsf{w}\!\uparrow\}\\ \mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{([\Phi, A], [\Psi, B])}{[\Psi, ([\Phi, A], B)]}}{[\Phi, \Psi, (A, B)]}\end{array} .$$

From Sequent Calculus to CoS – cont'd

Theorem 2.3.3. *For every derivation* $\begin{array}{c}\Sigma_1 \; \cdots \; \Sigma_h \\ \bigtriangledown \\ \Sigma\end{array}$ *in* $\mathsf{GS1p} + \mathsf{Cut}$ *there exists a derivation* $\begin{array}{c}(\underline{\Sigma_1}_S, \ldots, \underline{\Sigma_h}_S) \\ \Big\| \mathsf{SKSg} \setminus \{\mathsf{c}\!\uparrow, \mathsf{w}\!\uparrow\} \\ \underline{\Sigma}_S\end{array}$ *with the same number of cuts.*

... and since proofs are derivations of a specific form, these hold:

- Proofs (with cut) correspond to proofs in $\mathit{SKSg} \setminus \{\mathsf{c}\!\uparrow, \mathsf{w}\!\uparrow\}$
- Cut-free proofs correspond to proofs in $\mathit{SKSg} \setminus \{\mathsf{i}\!\uparrow, \mathsf{c}\!\uparrow, \mathsf{w}\!\uparrow\}$

From CoS to Sequent Calculus

Theorem 2.3.8. *For every derivation* $\begin{array}{c} Q \\ \Vert \; \mathsf{SKSg} \\ P \end{array}$ *there exists a derivation*

$\vdash \underline{Q}_G$

$\vdash \underline{P}_G$

in GS1p + Cut.

(And similarly, for proofs)

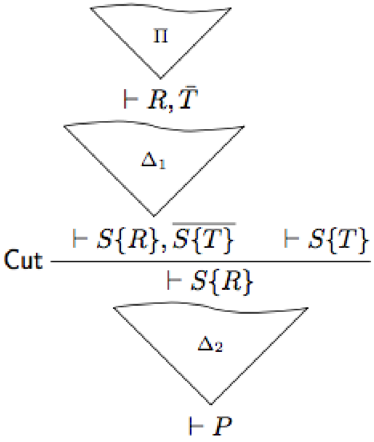
Sketch of proof

- Translate structures of *SKS* into formulae;
- Mimic deep inference within context in the sequent calculus;
- Proceed top-down, starting from the top-most rule, by induction on the length of the SKS derivation

From CoS to Sequent Calculus – cont'd

SKS derivation Δ (left) and corresponding construction (right)

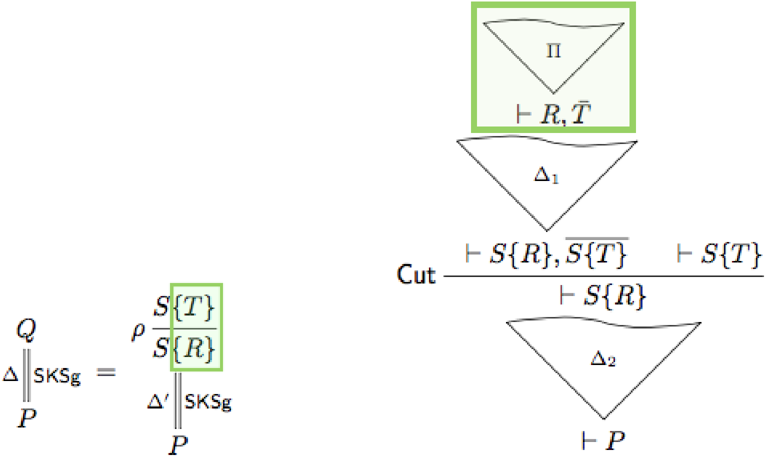
$$\begin{array}{c} Q \\ \Delta \Big\| \mathsf{SKSg} \\ P \end{array} \quad = \quad \begin{array}{c} \rho \dfrac{S\{T\}}{S\{R\}} \\ \Delta' \Big\| \mathsf{SKSg} \\ P \end{array}$$



From CoS to Sequent Calculus – cont'd

SKS derivation Δ (left) and corresponding construction (right)

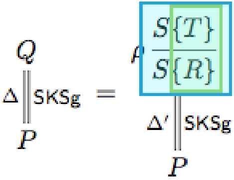
- Proof Π exists, specific to rule ρ ($T \Longrightarrow R$)

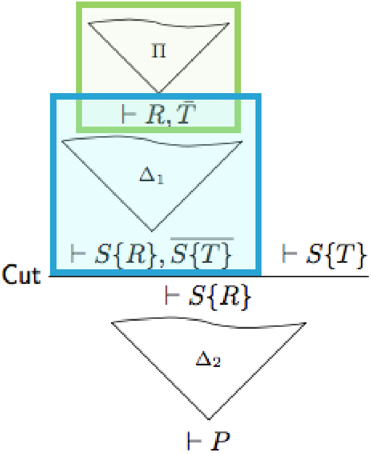


From CoS to Sequent Calculus – cont'd

SKS derivation Δ (left) and corresponding construction (right)

- Mimic the specific instance of ρ, in context $S\{\ \}$ (lemma)



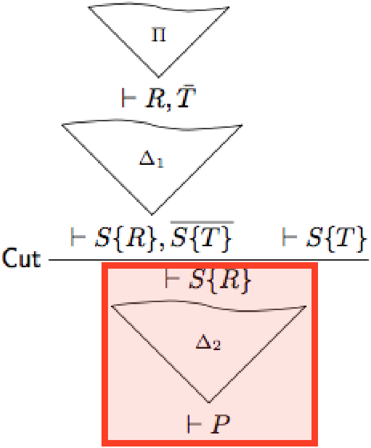


From CoS to Sequent Calculus – cont'd

KS derivation Δ (left) and corresponding construction (right)

- Inductive hypothesis

$$\Delta \left\| {}^{Q}_{P} \right. \mathsf{SKSg} \quad = \quad \rho \frac{S\{T\}}{\begin{array}{c} S\{R\} \\ \Delta' \Big\| \mathsf{SKSg} \\ P \end{array}}$$



From CoS to Sequent Calculus – cont'd

In particular, the up-rules will have these proofs associated:

$$\mathsf{i}\!\uparrow \frac{S(U,\bar{U})}{S\{\mathsf{f}\}} \quad , \qquad \mathsf{c}\!\uparrow \frac{S\{U\}}{S(U,U)} \quad , \qquad \mathsf{w}\!\uparrow \frac{S\{U\}}{S\{\mathsf{t}\}} \quad ,$$

$$\mathsf{RW}\,\dfrac{\mathsf{R}\vee\,\dfrac{\mathsf{Ax}\,\dfrac{}{\vdash U,\bar{U}}}{\vdash \bar{U}\vee U}}{\vdash \bot,\bar{U}\vee U}$$

$$\mathsf{RC}\,\dfrac{\mathsf{R}\wedge\,\dfrac{\mathsf{Ax}\,\dfrac{}{\vdash U,\bar{U}} \qquad \mathsf{Ax}\,\dfrac{}{\vdash U,\bar{U}}}{\vdash U\wedge U,\bar{U},\bar{U}}}{\vdash U\wedge U,\bar{U}}$$

$$\mathsf{RW}\,\dfrac{\top\,\dfrac{}{\vdash \top}}{\vdash \top,\bar{U}}$$

- *KSg* is *SKSg* without up-rules;
- Up-rules are admissible for *KSg*
  - A proof in *SKSg* is translated into one in $GSp1 + Cut$,
  - Cut-elimination holds: get a cut free proof in $GS1p$,
  - Translate back, it is a proof in *KSg*

From CoS to Sequent Calculus – cont'd

In particular, the up-rules will have these proofs associated:

$$\mathsf{i}\!\uparrow \frac{S(U,\bar{U})}{S\{\mathsf{f}\}} \;, \qquad \mathsf{c}\!\uparrow \frac{S\{U\}}{S(U,U)} \;, \qquad \mathsf{w}\!\uparrow \frac{S\{U\}}{S\{\mathsf{t}\}} \;,$$

$$\mathsf{RW}\frac{\mathsf{R}\vee\dfrac{\mathsf{Ax}\dfrac{}{\vdash U,\bar{U}}}{\vdash \bar{U}\vee U}}{\vdash \bot, \bar{U}\vee U}
\qquad
\mathsf{RC}\frac{\mathsf{R}\wedge\dfrac{\mathsf{Ax}\dfrac{}{\vdash U,\bar{U}} \quad \mathsf{Ax}\dfrac{}{\vdash U,\bar{U}}}{\vdash U\wedge U, \bar{U}, \bar{U}}}{\vdash U\wedge U, \bar{U}}
\qquad
\mathsf{RW}\frac{\top\dfrac{}{\vdash \top}}{\vdash \top, \bar{U}}$$

Two proof systems S_1 and S_2 are said

- (weakly) equivalent – for every proof of R in S_1 there is a proof of R in S_2, and viceversa;
- strongly equivalent – for every derivation from T to R in S_1, there is a derivation from T to R in S_2, and viceversa;
- e.g. *SKSg* and *KSg* are equivalent but not strongly equivalent

KSg – Remarks

$$\mathsf{i}{\downarrow}\,\frac{S\{\mathsf{t}\}}{S[R,\bar{R}]} \qquad \mathsf{s}\,\frac{S([R,T],U)}{S[(R,U),T]} \qquad \mathsf{w}{\downarrow}\,\frac{S\{\mathsf{f}\}}{S\{R\}} \qquad \mathsf{c}{\downarrow}\,\frac{S[R,R]}{S\{R\}}$$

- Cut-free sequent system: all rules fulfill subformula property;
- Down-fragment in deep inference: premisses of rules do not have *new* atoms that are not in the conclusion

KSg – Remarks

$$\mathsf{i}\!\downarrow \frac{S\{\mathsf{t}\}}{S[R,\bar{R}]} \qquad \mathsf{s}\,\frac{S([R,T],U)}{S[(R,U),T]} \qquad \mathsf{w}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{R\}} \qquad \mathsf{c}\!\downarrow \frac{S[R,R]}{S\{R\}}$$

- Notion of invertible rule of sequent calculus is imported:

  Definition 2.4.7. A rule ρ is *invertible* for a system $\mathscr{S}$ if for each instance $\rho\,\frac{V}{U}$ there is a derivation $\begin{array}{c}U\\ \Vert\mathscr{S}\\ V\end{array}$.

- .. and it is used to separate parts of the system (S' is the cnf of S)

  For every proof $\begin{array}{c}\overline{\Vert}\,\mathsf{SKSg}\\ S\end{array}$ *there is a proof* $\begin{array}{c}\overline{\Vert}\,\{\mathsf{i}\!\downarrow\}\\ S''\\ \Vert\,\{\mathsf{w}\!\downarrow\}\\ S'\\ \Vert\,\{\mathsf{s},\mathsf{c}\!\downarrow\}\\ S\end{array}$.

Locality via Atomic Rules - *SKS*

Objective – make interaction/weakening/contraction all atomic

In sequent calculus:

- atomic axiom may replace a general one, but making atomic a cut, given a general system, is not a free lunch.
- GS1p (with multiplicative $\wedge_R$) does not allow contraction to be made atomic:

$$\vdash (a \wedge b), (\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b})$$

In deep inference:

- Making interaction/cut and weakening (and dual) atomic is easy
- Making atomic contraction requires the *medial rule* (derivable in sequent calculus, but not as a rule):

$$\mathsf{m}\,\frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

$$\mathsf{c}\!\downarrow\frac{\mathsf{w}\!\downarrow\frac{\mathsf{w}\!\downarrow\frac{\mathsf{w}\!\downarrow\frac{\mathsf{w}\!\downarrow\frac{S[(R,U),(T,V)]}{S[(R,U),(T,[U,V])]}}{S[(R,U),([R,T],[U,V])]}}{S[(R,[U,V]),([R,T],[U,V])]}}{S[([R,T],[U,V]),([R,T],[U,V])]}}{S([R,T],[U,V])}$$

Locality via Atomic Rules - *SKS*

$$\text{ai}\!\downarrow \frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \qquad \text{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\mathsf{f}\}}$$

$$\text{s}\, \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\text{m}\, \frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

$$\text{aw}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{a\}} \qquad \text{aw}\!\uparrow \frac{S\{a\}}{S\{\mathsf{t}\}}$$

$$\text{ac}\!\downarrow \frac{S[a,a]}{S\{a\}} \qquad \text{ac}\!\uparrow \frac{S\{a\}}{S(a,a)}$$

Locality via Atomic Rules - *SKS* and *KS*

$$\mathsf{ai}{\downarrow}\ \frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \qquad \mathsf{ai}{\uparrow}\ \frac{S(a,\bar{a})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{m}\ \frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

$$\mathsf{aw}{\downarrow}\ \frac{S\{\mathsf{f}\}}{S\{a\}} \qquad \mathsf{aw}{\uparrow}\ \frac{S\{a\}}{S\{\mathsf{t}\}}$$

$$\mathsf{ac}{\downarrow}\ \frac{S[a,a]}{S\{a\}} \qquad \mathsf{ac}{\uparrow}\ \frac{S\{a\}}{S(a,a)}$$

Derivability of General Rules

Theorem 4.1.2. *The rules* $\mathsf{i}\downarrow$, $\mathsf{w}\downarrow$ *and* $\mathsf{c}\downarrow$ *are derivable for* $\{\mathsf{ai}\downarrow, \mathsf{s}\}$, $\{\mathsf{aw}\downarrow, \mathsf{s}\}$ *and* $\{\mathsf{ac}\downarrow, \mathsf{m}\}$, *respectively. Dually, the rules* $\mathsf{i}\uparrow$, $\mathsf{w}\uparrow$ *and* $\mathsf{c}\uparrow$ *are derivable for* $\{\mathsf{ai}\uparrow, \mathsf{s}\}$, $\{\mathsf{aw}\uparrow, \mathsf{s}\}$ *and* $\{\mathsf{ac}\uparrow, \mathsf{m}\}$, *respectively.*

Sketch of proof: cases for weakening, *R* not an atom (and dual rule)

$$\mathsf{w}\downarrow \frac{S\{\mathsf{f}\}}{S\{R\}}$$

$$= \frac{S\{\mathsf{f}\}}{\mathsf{s}\,\dfrac{S([\mathsf{t},\mathsf{t}],\mathsf{f})}{= \dfrac{S[\mathsf{t},(\mathsf{t},\mathsf{f})]}{S\{\mathsf{t}\}}}} \qquad = \frac{S\{\mathsf{f}\}}{\mathsf{w}\downarrow \dfrac{S[\mathsf{f},\mathsf{f}]}{\mathsf{w}\downarrow \dfrac{S[\mathsf{f},Q]}{S[P,Q]}}} \qquad = \frac{S\{\mathsf{f}\}}{\mathsf{w}\downarrow \dfrac{S(\mathsf{f},\mathsf{f})}{\mathsf{w}\downarrow \dfrac{S(\mathsf{f},Q)}{S(P,Q)}}}$$

whereas, for contraction, medial is needed in this case

$$\mathsf{m}\,\frac{S[(P,Q),(P,Q)]}{\mathsf{c}\downarrow \dfrac{S([P,P],[Q,Q])}{\mathsf{c}\downarrow \dfrac{S([P,P],Q)}{S(P,Q)}}}$$

Derivability of General Rules

- Therefore, *KS* and *KSg* are strongly equivalent.
- We may occasionally use general rules in KS, just as shorthand notation.
- (Atomic) contraction is related to sharing

On Design: Extension to first order

- Structures (Formulae) extended with quantifiers:

$$S ::= \mathsf{f} \mid \mathsf{t} \mid a \mid [\underbrace{S,\dots,S}_{>0}] \mid (\underbrace{S,\dots,S}_{>0}) \mid \bar{S} \mid \exists x S \mid \forall x S$$

- Syntactic equivalence on formulae are extended with

Variable Renaming	$\forall x R = \forall y R[x/y]$ $\exists x R = \exists y R[x/y]$	if y is not free in R
Vacuous Quantifier	$\forall y R = \exists y R = R$	if y is not free in R
Negation	$\overline{\exists x R} = \forall x \bar{R}$ $\overline{\forall x R} = \exists x \bar{R}$	

- Remark – the quantifier rules in *GS1* are

$$\mathsf{R}\exists \frac{\vdash \Phi, A[x/\tau]}{\vdash \Phi, \exists x A} \qquad \mathsf{R}\forall \frac{\vdash \Phi, A[x/y]}{\vdash \Phi, \forall x A}$$

Proviso: y is not free in the conclusion of $\mathsf{R}\forall$.

SKSgq – General, first order

$$\mathsf{i}\!\downarrow \frac{S\{\mathsf{t}\}}{S[R,\bar{R}]} \qquad \mathsf{i}\!\uparrow \frac{S(R,\bar{R})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\, \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{u}\!\downarrow \frac{S\{\forall x[R,T]\}}{S[\forall xR,\exists xT]} \qquad \mathsf{u}\!\uparrow \frac{S(\exists xR,\forall xT)}{S\{\exists x(R,T)\}}$$

$$\mathsf{w}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{R\}} \qquad \mathsf{w}\!\uparrow \frac{S\{R\}}{S\{\mathsf{t}\}}$$

$$\mathsf{c}\!\downarrow \frac{S[R,R]}{S\{R\}} \qquad \mathsf{c}\!\uparrow \frac{S\{R\}}{S(R,R)}$$

$$\mathsf{n}\!\downarrow \frac{S\{R[x/\tau]\}}{S\{\exists xR\}} \qquad \mathsf{n}\!\uparrow \frac{S\{\forall xR\}}{S\{R[x/\tau]\}}$$

SKSgq – General, first order

$$\mathsf{i}\!\downarrow \frac{S\{\mathsf{t}\}}{S[R,\bar{R}]} \qquad \mathsf{i}\!\uparrow \frac{S(R,\bar{R})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\, \frac{S([R,U],T)}{S[(R,T),U]}$$

R-Forall

$$\mathsf{u}\!\downarrow \frac{S\{\forall x[R,T]\}}{S[\forall xR,\exists xT]} \qquad \mathsf{u}\!\uparrow \frac{S(\exists xR,\forall xT)}{S\{\exists x(R,T)\}}$$

$$\mathsf{w}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{R\}} \qquad \mathsf{w}\!\uparrow \frac{S\{R\}}{S\{\mathsf{t}\}}$$

$$\mathsf{c}\!\downarrow \frac{S[R,R]}{S\{R\}} \qquad \mathsf{c}\!\uparrow \frac{S\{R\}}{S(R,R)}$$

R-Exists

$$\mathsf{n}\!\downarrow \frac{S\{R[x/\tau]\}}{S\{\exists xR\}} \qquad \mathsf{n}\!\uparrow \frac{S\{\forall xR\}}{S\{R[x/\tau]\}}$$

SKSgq – General, first order

- $u\downarrow$ – premiss implies conclusion (differently from GS1).

$$\mathsf{u}\!\downarrow \frac{\dfrac{S\{\forall x[R,T]\}}{S[\forall x R, \exists x T]}}{S[\forall x R, T]} \quad \text{if } x \text{ is not free in } T,$$

- $n\downarrow$ – the term τ is not required to be free for x in $S\{R\}$, i.e. there can be quantifiers in context $S\{\ \}$ that may capture variables in τ.
- Both rules seem more local
- Results of the propositional case are extended to the predicate case. In particular Deduction theorem, admissibility of up-fragment (and hence indirect proof of cut-elimination)

(*S*)*KSq* – Atomic, first order

Instances of contraction over quantified formulae – any interference with medial towards atomic contraction?

- Two more rules are needed (and their dual ones in the symmetric system);
- Strong equivalence on the first order *KSq* and *KSgq*

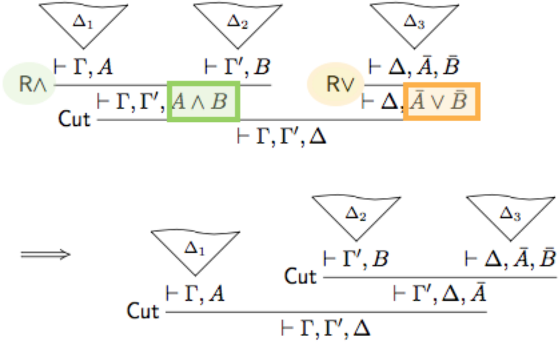
$$\mathsf{ai}\!\downarrow \frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \qquad \mathsf{aw}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{a\}} \qquad \mathsf{ac}\!\downarrow \frac{S[a,a]}{S\{a\}}$$

$$\mathsf{s}\,\frac{S([R,T],U)}{S[(R,U),T]} \qquad \mathsf{u}\!\downarrow \frac{S\{\forall x[R,T]\}}{S[\forall xR,\exists xT]} \qquad \mathsf{m}\,\frac{S[(R,T),(U,V)]}{S([R,U],[T,V])}$$

$$\mathsf{n}\!\downarrow \frac{S\{R[x/\tau]\}}{S\{\exists xR\}} \qquad \mathsf{l}_1\!\downarrow \frac{S[\exists xR,\exists xT]}{S\{\exists x[R,T]\}} \qquad \mathsf{l}_2\!\downarrow \frac{S[\forall xR,\forall xT]}{S\{\forall x[R,T]\}}$$

Reminder: Cut Elimination in Sequent Calculus

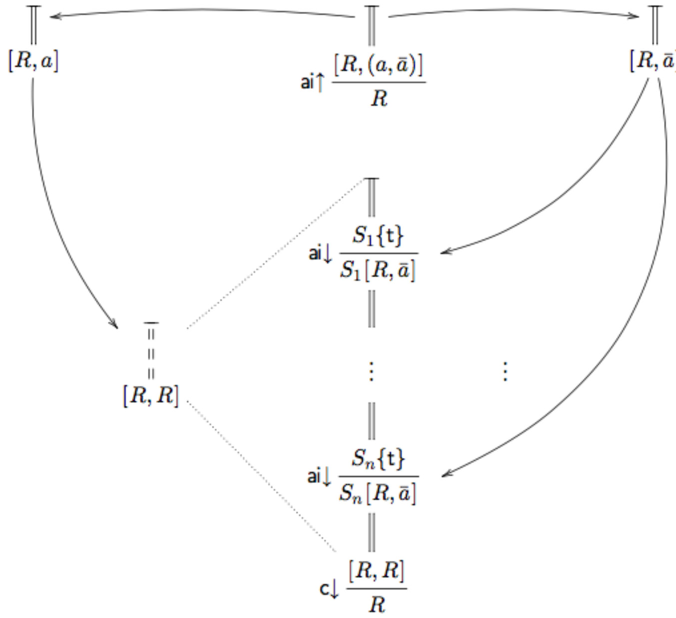
- Above the Cut (in a branching situation), two 'dual' logical rules operate on the cut formula (and its dual),
- just on their respective main connective.
- Restricting the cut rule to be atomic would help.
- This method cannot be adapted to deep inference so easily



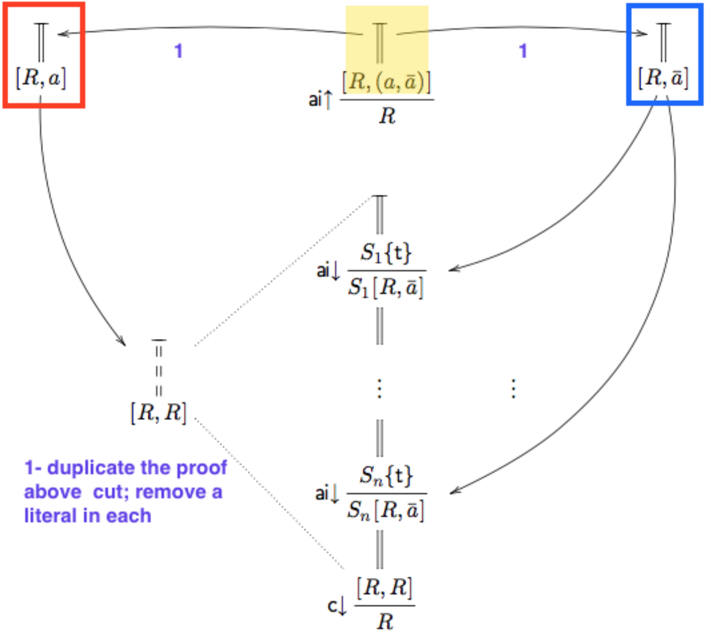
Cut Elimination in *SKS* - Idea (Propositional)

- This cut-elimination procedure is inspirational: a mixture between natural deduction and proper context rewriting
- Based mostly on works by Brünnler and Tiu, e.g. see [1].
- But it does not easily scale up to first order case.

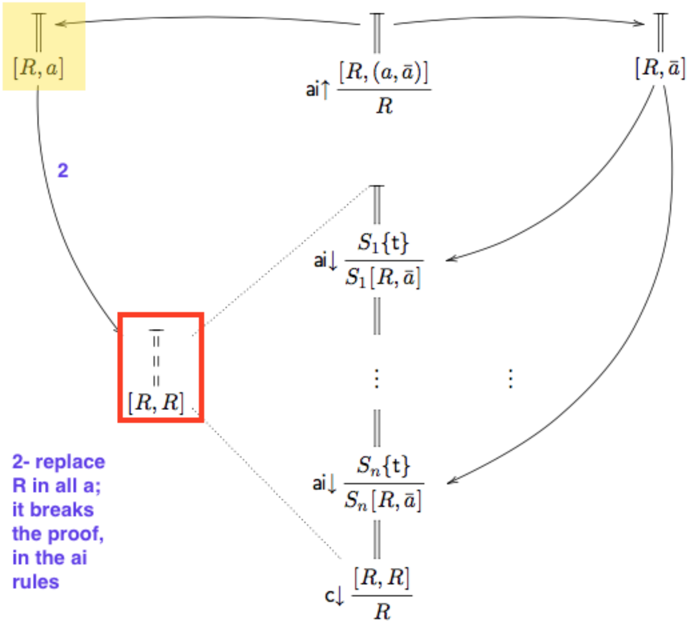
Cut Elimination in *SKS* - Idea



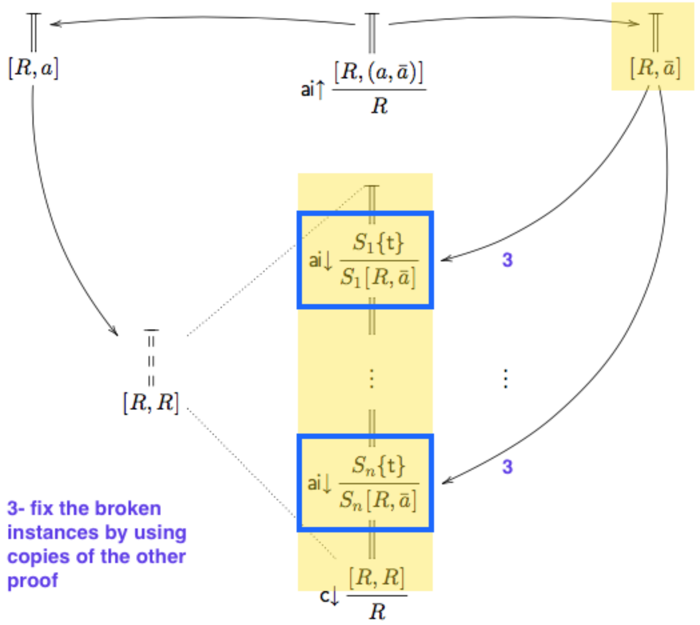
Cut Elimination in *SKS* - Idea



Cut Elimination in *SKS* - Idea

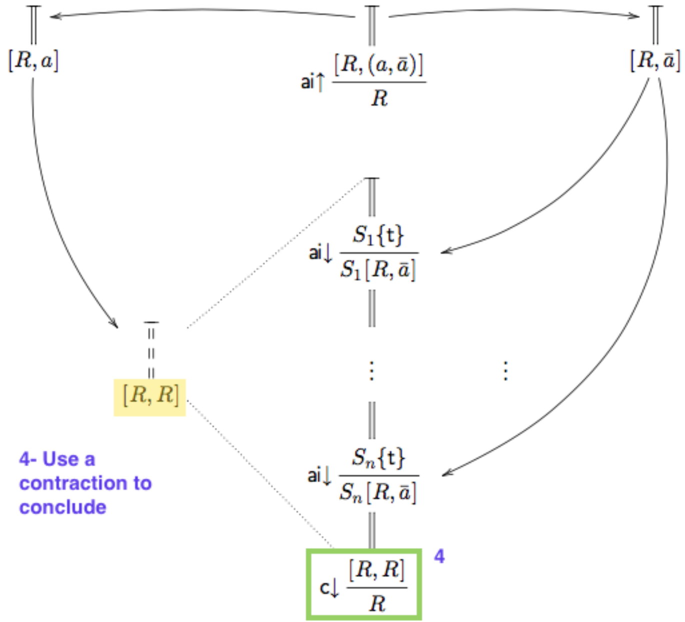


Cut Elimination in *SKS* - Idea



3- fix the broken instances by using copies of the other proof

Cut Elimination in *SKS* - Idea



Lemmata for the Proof of Cut Elim'on in *SKS*

- Each rule ρ in SKS is derivable for $i\downarrow, i\uparrow, s$ and the dual of ρ.
  1–Transform the original *SKS* into one in $KS \cup i\uparrow$ (or atomic)
- Atomic cut is derivable for shallow atomic cut (below, left) and *s*:

$$\mathsf{ai}\!\uparrow \frac{[S,(a,\bar{a})]}{S}$$

$$\mathsf{ai}\!\uparrow \frac{\dfrac{S([R,(a,\bar{a})],T)}{S([R,\mathsf{f}],T)}}{=\;\dfrac{}{S(R,T)}} \quad \leadsto \quad \mathsf{s}\,\frac{S([R,(a,\bar{a})],T)}{\mathsf{ai}\!\uparrow \dfrac{S[(R,T),(a,\bar{a})]}{= \dfrac{S[(R,T),\mathsf{f}]}{S(R,T)}}}$$

  2–Deal only with shallow atomic cut

- Any proof of $T\{a\}$ in *KS* can be transformed into one of $T\{t\}$ in *KS*.
  Trace-replace the occurrences of *a*, bottom-up in a proof – e.g.

$$\mathsf{ai}\!\downarrow \frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \quad \leadsto \quad = \frac{S\{\mathsf{t}\}}{\mathsf{aw}\!\downarrow \dfrac{S[\mathsf{t},\mathsf{f}]}{S[\mathsf{t},\bar{a}]}}$$

  3–Generate this way the two initial copies of proofs above the cut

Cut Elimination in *SKS*

Start with a transformed proof: only shallow atomic cuts as up-rule.
Consider the topmost cut, and generate the two copies of the proof above the cut (use a/t and $\bar{a}/t$):

$$
\begin{array}{c}
\Pi \Big\Vert \mathsf{KS} \\
\mathsf{ai}\!\uparrow \dfrac{[R,(a,\bar{a})]}{R} \\
\Delta \Big\Vert \mathsf{KS} \cup \{\mathsf{ai}\!\uparrow\} \\
T
\end{array}
\qquad\qquad
\begin{array}{c}
\Pi_1 \Big\Vert \mathsf{KS} \\
[R,a]
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\Pi_2 \Big\Vert \mathsf{KS} \\
[R,\bar{a}]
\end{array}
$$

Bottom-up in Π_1 replace a/R – no effect if a is in the context or in s, m.
Otherwise, fix it (left) to paste it for the cut-eliminated proof (right),

$$
\mathsf{ac}\!\downarrow \frac{S[a,a]}{S\{a\}} \;\leadsto\; \mathsf{c}\!\downarrow \frac{S[R,R]}{S\{R\}}
$$

$$
\mathsf{aw}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{a\}} \;\leadsto\; \mathsf{w}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{R\}}
\qquad
\mathsf{ai}\!\downarrow \frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \;\leadsto\;
\begin{array}{c}
S\{\mathsf{t}\} \\
S\{\Pi_2\} \Big\Vert \mathsf{KS} \\
S[R,\bar{a}]
\end{array}
\qquad\qquad
\begin{array}{c}
\Pi_3 \Big\Vert \mathsf{KS} \\
\mathsf{c}\!\downarrow \dfrac{[R,R]}{R} \\
\Delta \Big\Vert \mathsf{KS} \cup \{\mathsf{ai}\!\downarrow\} \\
T
\end{array}
$$

Cut Elimination in *SKS*

Start with a transformed proof: only shallow atomic cuts as up-rule.
Consider the topmost cut, and generate the two copies of the proof above the cut (use a/t and $\bar{a}/t$):

$$
\begin{array}{c}
\Pi \Vert \mathsf{KS} \\
\mathsf{ai}\uparrow \dfrac{[R,(a,\bar{a})]}{R} \\
\Delta \Vert \mathsf{KS}\cup\{\mathsf{ai}\uparrow\} \\
T
\end{array}
\qquad\qquad
\begin{array}{c}
\Pi_1 \Vert \mathsf{KS} \\
{[R,a]}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\Pi_2 \Vert \mathsf{KS} \\
{[R,\bar{a}]}
\end{array}
$$

Bottom-up in Π_1 replace a/R – no effect if a is in the context or in s, m.
Otherwise, fix it (left) to paste it for the cut-eliminated proof (right),

$$
\mathsf{ac}\downarrow \frac{S[a,a]}{S\{a\}} \;\leadsto\; \mathsf{c}\downarrow \frac{S[R,R]}{S\{R\}}
$$

$$
\mathsf{aw}\downarrow \frac{S\{\mathsf{f}\}}{S\{a\}} \;\leadsto\; \mathsf{w}\downarrow \frac{S\{\mathsf{f}\}}{S\{R\}}
\qquad
\mathsf{ai}\downarrow \frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \;\leadsto\;
\begin{array}{c}
S\{\mathsf{t}\} \\
S\{\Pi_2\} \Vert \mathsf{KS} \\
S[R,\bar{a}]
\end{array}
\qquad
\begin{array}{c}
\Pi_3 \Vert \mathsf{KS} \\
\mathsf{c}\downarrow \dfrac{[R,R]}{R} \\
\Delta \Vert \mathsf{KS}\cup\{\mathsf{ai}\downarrow\} \\
T
\end{array}
$$

Decomposition and Normal Forms

Studying the permutability of rules in *KS*/*SKS* allows the discovery of ways to decompose proofs/derivations.

Definition 7.1.1. A rule ρ *permutes over* a rule π (or π *permutes under* ρ) if for every derivation $\pi \frac{T}{\rho \frac{U}{R}}$ there is a derivation $\rho \frac{T}{\pi \frac{V}{R}}$ for some formula V.

(... various transformations, for different logics, but with some resemblances)

Decomposition and Normal Forms

Example 1: separating cut and interaction

$$\text{For every derivation } \begin{array}{c} T \\ \Vert\,\mathsf{SKS} \\ R \end{array} \text{ there is a derivation } \begin{array}{c} T \\ \Vert\,\{\mathsf{ai}\downarrow\} \\ V \\ \Vert\,\mathsf{SKS}\setminus\{\mathsf{ai}\downarrow,\mathsf{ai}\uparrow\} \\ U \\ \Vert\,\{\mathsf{ai}\uparrow\} \\ R \end{array} .$$

Example 2: separate contraction (not possible in sequent calculus).

$$\text{For every proof } \begin{array}{c} \overline{\Vert}\,\mathsf{KS} \\ S \end{array} \text{ there is a proof } \begin{array}{c} \overline{\Vert}\,\mathsf{KS}\setminus\{\mathsf{ac}\downarrow\} \\ S' \\ \Vert\,\{\mathsf{ac}\downarrow\} \\ S \end{array} .$$

Example 3: separate weakening in a proof.

$$\text{For every proof } \begin{array}{c} \overline{\Vert}\,\mathsf{KS} \\ S \end{array} \text{ there is a proof } \begin{array}{c} \overline{\Vert}\,\mathsf{KS}\setminus\{\mathsf{aw}\downarrow\} \\ S' \\ \Vert\,\{\mathsf{aw}\downarrow\} \\ S \end{array} .$$

Some Remarks

- Some of these decompositions entail elimination of cuts;
- They can be used for an interpolation theorem;
- The 'layering' of rules application (decomposition) is informative and may be used to guide the proof-search process;
- It can also support incremental design of extensions of the system, with new connectives;
- Choices in the design of inference rules may impact on other theorems, for example Herbrand's theorem (a good overview is in Ralph's PhD thesis [5])

(.. just to mention a few..)

Some Proposed Activities

- Is it possible to build a derivation with premiss c and conclusion $[(a\ a)\ (b\ c)\ \bar{a}\ (\bar{b}\ c)]$ in SKS? And in $KS \cup \{i\uparrow\}$?
- Are $KS \cup \{i\uparrow\}$ and $KS \cup \{c\uparrow\}$ strongly equivalent? Are they equivalent? Are they equivalent to SKS?
- You might try and prove some of the case analyses that establish the correspondence between derivations in $GS1p + Cut$ and in SKS.
- In the translation from CoS to sequent calculus, we have mentioned (but not even sketched) the need of a lemma to mimic the deep application of an inference rule ρ in a context $S\{\ \}$. You might like to reconstruct that proof.
- Complete the proof that $c\uparrow$ is derivable in $\{ac\uparrow, m\}$.

Deep inference web site: `http://alessio.guglielmi.name/res/cos/`

References I

[1] Kai Brünnler (2004): *Deep Inference and Symmetry in Classical Proofs.*
Logos Verlag, Berlin.
Available at `http://cs.bath.ac.uk/ag/kai/phd.pdf`.

[2] Paola Bruscoli (2002): *A Purely Logical Account of Sequentiality in Proof Search.*
In Peter J. Stuckey, editor: *Logic Programming, 18th International Conference (ICLP)*, *Lecture Notes in Computer Science* 2401, Springer-Verlag, pp. 302–316, doi:10.1007/3-540-45619-8_21.
Available at `http://cs.bath.ac.uk/pb/bvl/bvl.pdf`.

[3] Alessio Guglielmi (2007): *A System of Interaction and Structure.*
ACM Transactions on Computational Logic 8(1), pp. 1:1–64, doi:10.1145/1182613.1182614.
Available at `http://cs.bath.ac.uk/ag/p/SystIntStr.pdf`.

[4] Alessio Guglielmi (2015): *Deep Inference.*
In David Delahaye & Bruno Woltzenlogel Paleo, editors: *All About Proofs, Proofs for All*, *Mathematical Logic and Foundations* 55, College Publications, pp. 164–172.
Available at `http://cs.bath.ac.uk/ag/p/DI.pdf`.

[5] Benjamin Ralph (2019): *Modular Normalisation of Classical Proofs.*
Ph.D. thesis, University of Bath.
Available at `http://cs.bath.ac.uk/ag/br/phd.pdf`.

[6] Alwen Tiu (2006): *A System of Interaction and Structure II: The Need for Deep Inference.*
Logical Methods in Computer Science 2(2), pp. 4:1–24, doi:10.2168/LMCS-2(2:4)2006.
Available at `https://arxiv.org/pdf/cs/0512036.pdf`.