Three Lectures on Structural Proof Theory

1 – Traditional Formalisms

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Course Notes
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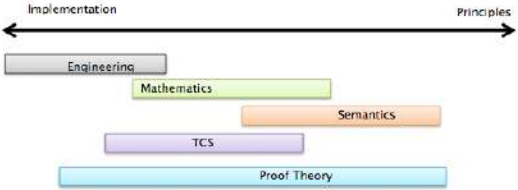
Plan of the Course

- Purpose: Studying proofs and proofs manipulations, with an evolutionary spirit: rethinking the traditional methodologies from the computer science perspective;
- Focus is on aspects of design: (of formalisms and systems therein) how it affects the structure of proofs and the proof theory.
- Scope: this course is limited to intuitionistic and classical logic only and ultimately addresses
  - Analyticity, Cut elimination, Herbrand theorem;
  - The move to deep inference and its theorems;
- Activities (and readings supporting them) are proposed throughout the slides and appear **in this colour**. These (extensive) slides are compact but rich enough to qualify as course notes.

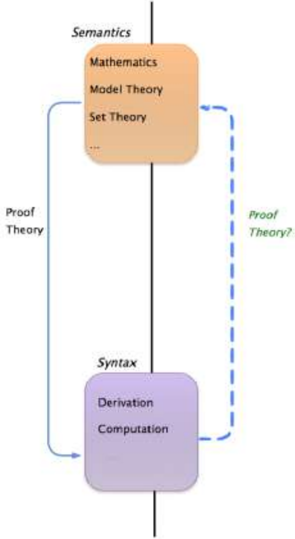
Proof Theory for and from Computer Science



Proof theory is for computer science

- a foundational tool:
  it connects semantics with syntax, it is about consistent formal system and semantics might be absent
- a means for producing practical results:
  it studies finitary systems that in principle are implementable (e.g. functional and logic programming)
- and it interacts with Category Theory that is the other major founding tool in computing.

Proof Theory for and from Computer Science



Mathematics

1900–1928 Hilbert:

Can we found mathematics in a finitary way?

1931 Gödel:

No...

1935 Gentzen:

.. but we can 'tame' infinity

Computer Science

What are the fundamental properties of computation?

Is computation inherently chaotic?

Can proof theory help, maybe with substructural logics?
(Example: typed lambda-calculus)

When are two proofs the same?

To what extent do formal proof transformations inform the problem of equality of proofs?

"Cost" of these transformations?

Re-thinking the Tradition

"Traditional" proof theoretic methods

- ... are essentially linked to sequential computation (notion of main connective), whereas we probably want to understand distributed computation;
- ... originated within classical/intuitionistic logics, whereas we may need different ones, typically linear and substructural logics;
- ... are too rigid when used for linear and substructural logics, for they hide to the observer the possibility of relevant transformations of proof theoretic interest.

"Deep inference" methods proposed to address

- linearity, atomicity, locality, sharing, analyticity and compositionality,
- with a complexity-aware mindset,
- and extends and expands the traditional methods.

Outline for Today

Survey of Traditional Formalisms and Systems
Subformula Property and Cut Rule in Sequent Calculi
Classical as Extension of Intuitionistic
About Cut Elimination

Hilbert-Tarski (or Frege) Style - Classical

(HT$_1$) $A \supset (B \supset A)$,

(HT$_2$) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$,

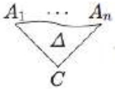
(HT$_3$) $(\neg B \supset \neg A) \supset ((\neg B \supset A) \supset B)$,

(HT$_4$) $\forall x.A \supset A[t/x]$,

(HT$_5$) $\forall x.(A \supset B) \supset (A \supset \forall x.B)$,

$$\text{mp}\ \frac{A \quad A \supset B}{B} \qquad \text{gen}\ \frac{A}{\forall x.A}$$

- Deduction/Proof Δ (a tree) of theorem C (conclusion), from hypotheses A_i (instances of axioms), built by applying inference rules (mp and gen):



- Proof of $A \supset A$:

$$\text{mp}\ \dfrac{\text{mp}\ \dfrac{[A \supset (\overbrace{(A \supset A)}^{B} \supset \overbrace{A}^{C})] \supset [(A \supset \overbrace{(A \supset A)}^{B}) \supset (A \supset \overbrace{A}^{C})] \qquad A \supset (\overbrace{(A \supset A)}^{B} \supset \overbrace{A}^{C})}{(A \supset (A \supset A)) \supset (A \supset A)} \qquad A \supset (A \supset A)}{A \supset A}$$

Axiomatic Systems - Some Interim Activities

- Is $\forall x.(\forall x.A \supset A)$ a theorem that can be proven in the given system?
- Another axiomatic system (propositional), with the same inference rules:

$$
\begin{array}{l}
(HT_1) \\
(HT_2) \\
(A \supset (B \supset C)) \supset (B \supset (A \supset C)) \\
(A \supset B) \supset (\neg B \supset \neg A) \\
\neg\neg A \supset A \\
A \supset \neg\neg A
\end{array}
$$

Do the two axiomatisations prove the same theorems? Would more initial axiom schemes be an advantage in finding the proof?

Natural Deduction

- introduction and elimination rules for each logical connective;
- notion of proof quite informal (it will then become 'derivability' in a given proof system - Gentzen)
- introduction rules: give BHK explanations in terms of direct provability
  - A d.p. of $A\&B$ ($A \vee B$) consists of proofs of A and B (A or B)
  - A d.p of $A \supset B$ consists of a proof of the B from the *assumption* that there is a proof of A (and assumption then discarded, written as [.])
  - A d.p of $\perp$ is impossible

$$\frac{A \quad B}{A\&B}\,\&I \qquad \frac{A}{A \vee B}\,\vee I1 \qquad \frac{B}{A \vee B}\,\vee I2 \qquad \frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \supset B}\,\supset I$$

- Note: Some rules are just one-step, but the 'dotted' one traverses the deduction

Natural Deduction-cont'd

- elimination rules: obtained from introduction rules by an inversion principle. They have propositions formed by logical constants as major premiss, and derive their consequences.
- Prawitz' inversion principle ('65) *The conclusion of an elimination rule with major premiss $A * B$ is already contained in the assumption used to derive $A * B$ from the $*_I$, together with the minor premiss of the rule.*
  - *i.e.* no gain, by introducing and then eliminating a connective in providing an explanation
  - But the $*_E$ rule is not uniquely determined.
- Generalised inversion principle *Whatever follows from the direct grounds for deriving a proposition must follow from that proposition.*

Natural Deduction-cont'd

rule	*combine*	*conversion*
$\dfrac{A\&B \qquad \begin{array}{c}[A,B]\\ \vdots\\ C\end{array}}{C}\,\&E$	$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array} \quad \begin{array}{c}\vdots\\ B\end{array}}{A\&B}\,\&I \qquad \begin{array}{c}[A,B]\\ \vdots\\ C\end{array}}{C}\,\&E$	$\begin{array}{c}\begin{array}{cc}\vdots & \vdots\\ A & B\end{array}\\ \vdots\\ C\end{array}$
$\dfrac{A \vee B \qquad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}\,\vee E$	$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{A \vee B}\,\vee I1 \qquad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}\,\vee E$	$\begin{array}{c}\vdots\\ A\\ \vdots\\ C\end{array}$
$\dfrac{A \supset B \qquad A \qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}\,\supset E$	$\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \supset B}\,\supset I \qquad \begin{array}{c}\vdots\\ A\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}\,\supset E$	$\begin{array}{c}\vdots\\ A\\ \vdots\\ B\\ \vdots\\ C\end{array}$

Natural Deduction - Assumptions Management

Management is needed, when discharging assumptions (and sometimes we must leave them open even if they could be discharged!)

- ND displays only open assumptions that are active in the rule:

$$\frac{[A]^1}{A \supset A} \supset_I^1$$

- When A is part of a larger context of assumptions, passive at this stage, that context will have to reflect that A has been discharged.

Natural Deduction - Assumptions Management

- Reading top-down, the first inference is justified because we work with sets of assumptions, i.e. $A - B = A$. This is a vacuous discharge of B

$$\dfrac{\dfrac{[A]^1}{B \supset A}\supset_I}{A \supset (B \supset A)}\supset_I^1$$

- When $B = A$ the deduction is not even correct. Other principles may handle this (the unique discharge principle); yet, the design is not very nice.

Natural Deduction - Assumptions Management

- Multiple discharge of *A*, at one rule, but there are two separate assumptions:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A \supset (A \supset B)]^2 \quad [A]^1}{A \supset B}\supset_E \qquad [A]^1}{B}\supset_E}{A \supset B}\supset_I^1}{(A \supset (A \supset B)) \supset (A \supset B)}\supset_I^2$$

- We have to traverse the deduction to chase those locations (non-locality in deductions)

The sequent style presentation of natural deduction by Gentzen addresses **all** these problems.

Natural Deduction (sequent style presentation)

- $\Gamma \vdash A$ sequent
- Γ finite set $\{x_1 : A_1, \ldots, x_n : A_n\}$, where x_i are pairwise disjoint labels (to keep track of open/closed assumptions); A, A_i formulae
- Notation: $\Gamma, x : A$ stands for set-union, provided that x is a pairwise-disjoint label from those in Γ
- Intuitionistic $N_I^{\supset,\wedge,\vee,\bot}$

Axiom

$$\Gamma, x{:}A \vdash A$$

Introduction Rules

$$\supset_I \frac{\Gamma, x{:}A \vdash B}{\Gamma \vdash A \supset B}$$

$$\wedge_I \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\vee_{IL} \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \vee_{IR} \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

Elimination Rules

$$\supset_E \frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

$$\wedge_{EL} \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \wedge_{ER} \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

$$\vee_E \frac{\Gamma \vdash A \vee B \quad \Gamma, x{:}A \vdash C \quad \Gamma, y{:}B \vdash C}{\Gamma \vdash C}$$

$$\bot_E \frac{\Gamma \vdash \bot}{\Gamma \vdash A} \text{ where } A \neq \bot$$

Natural Deduction (sequent style presentation)

- $\Gamma \vdash A$ sequent
- Γ finite set $\{x_1 : A_1, \ldots, x_n : A_n\}$, where x_i are pairwise disjoint labels
- Notation: $\Gamma, x : A$ stands for set-union, provided that x is a pairwise-disjoint label from those in Γ
- Intuitionistic $N_I^{\supset,\wedge,\vee,\bot}$ and classical $N_C^{\supset,\wedge,\vee,\bot}$ $(\neg A \doteq A \supset \bot)$

Axiom

$$\Gamma, x{:}\, A \vdash A$$

Introduction Rules

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$$\text{bc} \frac{\Gamma, x{:}\, \neg A \vdash \bot}{\Gamma \vdash A} \qquad \bot_E \frac{\Gamma \vdash \bot}{\Gamma \vdash A} \text{ where } A \neq \bot$$

Example: proofs of HT_I in $N_I^{\supset,\wedge,\vee,\bot}$

$$\supset_I \dfrac{\supset_I \dfrac{x\colon A, y\colon B \vdash A}{x\colon A \vdash B \supset A}}{\vdash A \supset (B \supset A)}$$

$$\supset_I \dfrac{\supset_I \dfrac{\wedge_{EL} \dfrac{\wedge_I \dfrac{x\colon A, y\colon B \vdash A \quad x\colon A, y\colon B \vdash B}{x\colon A, y\colon B \vdash A \wedge B}}{x\colon A, y\colon B \vdash A}}{x\colon A \vdash B \supset A}}{\vdash A \supset (B \supset A)}$$

$N_I^{\supset,\wedge,\vee,\bot}$

Axiom

$$\Gamma, x\colon A \vdash A$$

Introduction Rules

$$\supset_I \dfrac{\Gamma, x\colon A \vdash B}{\Gamma \vdash A \supset B}$$

$$\wedge_I \dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\vee_{IL} \dfrac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \vee_{IR} \dfrac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

Elimination Rules

$$\supset_E \dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

$$\wedge_{EL} \dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \wedge_{ER} \dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

$$\vee_E \dfrac{\Gamma \vdash A \vee B \quad \Gamma, x\colon A \vdash C \quad \Gamma, y\colon B \vdash C}{\Gamma \vdash C}$$

$$\bot_E \dfrac{\Gamma \vdash \bot}{\Gamma \vdash A} \text{ where } A \neq \bot$$

Example: proof of HT_3 in $N_C^{\supset,\wedge,\vee,\bot}$

$$
\supset_I \dfrac{\supset_I \dfrac{\text{bc}\ \dfrac{\supset_E \dfrac{\supset_E \dfrac{\Gamma \vdash \neg B \supset \neg A \quad \Gamma \vdash \neg B}{\Gamma \vdash \neg A} \qquad \supset_E \dfrac{\Gamma \vdash \neg B \supset A \quad \Gamma \vdash \neg B}{\Gamma \vdash A}}{x{:}\, \neg B \supset \neg A, y{:}\, \neg B \supset A, z{:}\, \neg B \vdash \bot}}{x{:}\, \neg B \supset \neg A, y{:}\, \neg B \supset A \vdash B}}{x{:}\, \neg B \supset \neg A \vdash (\neg B \supset A) \supset B}}{\vdash (\neg B \supset \neg A) \supset ((\neg B \supset A) \supset B)}
$$

where $\Gamma = \{x{:}\, \neg B \supset \neg A, y{:}\, \neg B \supset A, z{:}\, \neg B\}$

$N_C^{\supset,\wedge,\vee,\bot}$

Axiom

$$\Gamma, x{:}\, A \vdash A$$

Introduction Rules

$$\supset_I \dfrac{\Gamma, x{:}\, A \vdash B}{\Gamma \vdash A \supset B}$$

$$\wedge_I \dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\vee_{IL} \dfrac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \vee_{IR} \dfrac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

Elimination Rules

$$\supset_E \dfrac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

$$\wedge_{EL} \dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \wedge_{ER} \dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

$$\vee_E \dfrac{\Gamma \vdash A \vee B \quad \Gamma, x{:}\, A \vdash C \quad \Gamma, y{:}\, B \vdash C}{\Gamma \vdash C}$$

$$\text{bc}\ \dfrac{\Gamma, x{:}\, \neg A \vdash \bot}{\Gamma \vdash A} \qquad \bot_E \dfrac{\Gamma \vdash \bot}{\Gamma \vdash A} \ \text{where } A \neq \bot$$

Some Remarks

- Asymmetric sequents: only one formula on the right, also for classical logic?
- $\vee_E$ is the only rule with three premises and contains a spurious C not related to the disjunction;
- rules $\supset_I$ and $\vee_E$ require a global view of the context to discharge the assumptions
- .. and besides negation is treated in an indirect way

Axiom

$$\Gamma, x{:}A \vdash A$$

Introduction Rules

$$\supset_I \frac{\Gamma, x{:}A \vdash B}{\Gamma \vdash A \supset B}$$

$$\wedge_I \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\vee_{IL} \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \vee_{IR} \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

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$$bc \frac{\Gamma, x{:}\neg A \vdash \bot}{\Gamma \vdash A} \qquad \bot_E \frac{\Gamma \vdash \bot}{\Gamma \vdash A} \text{ where } A \neq \bot$$

Natural Deduction – Some Interim Activities

- Consider the examples illustrated when discussing assumption management in natural deduction. Reconstruct those cases using sequent style presentations.
- All that is provable in N_I is provable in N_C. How to prove it?
- Proving $A \equiv B$ means proving two implications. Can you prove that $(A \vee \neg B) \equiv \neg(\neg A \wedge B)$ in N_I?
- .. and in N_C?
- N_I is sufficient to prove the Curry-Howard isomorphism between Natural Deduction and simply typed lambda-calculus, isn't?

Sequent Calculus: Intuitionistic

Drop the labelling mechanism of natural deduction, the system becomes more symmetric

- multiset Γ (and Δ) in sequents: $\Gamma \vdash A$ (and $\Gamma \vdash \Delta$, classical)
- requires a contraction rule (structural) to handle multisets;
- Left/Right rules rather than Intro/Elim rules

Axiom

$$A, \Gamma \vdash A$$

Structural Rule

$$>_{\mathsf{L}} \frac{A, A, \Gamma \vdash C}{A, \Gamma \vdash C}$$

Left Rules

$$\supset_{\mathsf{L}} \frac{\Gamma \vdash A \quad B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C}$$

$$\wedge_{\mathsf{L}} \frac{A, B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C}$$

$$\vee_{\mathsf{L}} \frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \vee B, \Gamma \vdash C}$$

Right Rules

$$\supset_{\mathsf{R}} \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B}$$

$$\wedge_{\mathsf{R}} \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\vee_{\mathsf{RL}} \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \vee_{\mathsf{RR}} \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

$$\bot_{\mathsf{R}} \frac{\Gamma \vdash \bot}{\Gamma \vdash A} \text{ where } A \neq \bot$$

Fig. 6 *Intuitionistic Propositional Sequent Calculus* $\mathsf{G}_{\mathsf{i}}^{\supset, \wedge, \vee, \bot}$

A Glance at the Design..

Axiom

$$\Gamma, x{:}\, A \vdash A$$

Introduction Rules

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$$\bot_E \frac{\Gamma \vdash \bot}{\Gamma \vdash A} \text{ where } A \neq \bot$$

Axiom

$$A, \Gamma \vdash A$$

Structural Rule

$$>_L \frac{A, A, \Gamma \vdash C}{A, \Gamma \vdash C}$$

Left Rules

$$\supset_L \frac{\Gamma \vdash A \quad B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C}$$

$$\wedge_L \frac{A, B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C}$$

$$\vee_L \frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \vee B, \Gamma \vdash C}$$

Right Rules

$$\supset_R \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B}$$

$$\wedge_R \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\vee_{RL} \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \vee_{RR} \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

$$\bot_R \frac{\Gamma \vdash \bot}{\Gamma \vdash A} \text{ where } A \neq \bot$$

Subformula Property and Cut rule

Axiom

$$\Gamma, x{:}A \vdash A$$

Introduction Rules

$$\supset_I \frac{\Gamma, x{:}A \vdash B}{\Gamma \vdash A \supset B}$$

$$\wedge_I \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\vee_{IL} \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \vee_{IR} \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

Elimination Rules

$$\supset_E \frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

$$\wedge_{EL} \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \wedge_{ER} \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

$$\vee_E \frac{\Gamma \vdash A \vee B \quad \Gamma, x{:}A \vdash C \quad \Gamma, y{:}B \vdash C}{\Gamma \vdash C}$$

$$\bot_E \frac{\Gamma \vdash \bot}{\Gamma \vdash A} \text{ where } A \neq \bot$$

Axiom

$$A, \Gamma \vdash A$$

Structural Rule

$$>_L \frac{A, A, \Gamma \vdash C}{A, \Gamma \vdash C}$$

Left Rules

$$\supset_L \frac{\Gamma \vdash A \quad B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C}$$

$$\wedge_L \frac{A, B, \Gamma \vdash C}{A \wedge B, \Gamma \vdash C}$$

$$\vee_L \frac{A, \Gamma \vdash C \quad B, \Gamma \vdash C}{A \vee B, \Gamma \vdash C}$$

Right Rules

$$\supset_R \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B}$$

$$\wedge_R \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\vee_{RL} \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \vee_{RR} \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

$$\bot_R \frac{\Gamma \vdash \bot}{\Gamma \vdash A} \text{ where } A \neq \bot$$

Cut

$$\bowtie \frac{\Gamma \vdash A \quad A, \Gamma \vdash C}{\Gamma \vdash C}$$

Subformula Property and Cut rule

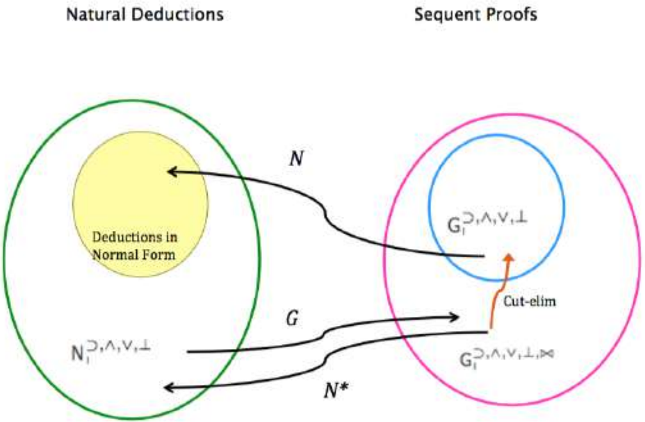
- Subformula Property: only subformulae of the conclusion should be in the premisses of a rule.
- Cut rule: no subformula property (not a finitary rule)

Cut

$$\bowtie \frac{\Gamma \vdash A \quad A, \Gamma \vdash C}{\Gamma \vdash C}$$

- Useful, when dealing with semantics or to relate different systems for the same logic, possibly in different formalisms; bad for automated deduction;
- Cut-elimination i.e. the cut is admissible: the other rules are complete for the system (possible, if the system is well designed)
- Contraction is needed, or some intuitionistic theorems would not be provable in $G_I^{\supset,\wedge,\vee,\perp}$ (also needed in proof of cut-elimination).

Relations among Systems



Cut-elimination holds: $G_I^{\supset,\wedge,\vee,\bot}$ and $G_I^{\supset,\wedge,\vee,\bot,\bowtie}$ prove the same theorems

Natural Deduction in Sequent Presentation – Some Interim Activities

- Let $\neg A$ be a shorthand for $A \supset \bot$. Try to prove $\neg\neg(A \vee \neg A)$ in $G_I^{\supset,\wedge,\vee,\bot}$ without using contraction.
- Full details of the translations N, N^*, G and cut elimination are available in Gallier's notes. Deductions in normal form correspond to beta-reductions; now we have cut-elimination as computation. Can you see such correspondence?
- Cut-free systems better support proof-search as computation (provided that non-determinism is tamed and there is a nice operational semantics). We agree on this, don't we?

Extension to Classical (and 1st order): $G_C^{\supset,\wedge,\vee,\neg,\forall,\exists}$

Introducing more symmetries:

- Multisets also on the right side of the sequent: $\Gamma \vdash \Delta$
  .. at 'fat' axiom
- Contraction also on the right side.
  On the left is not really needed, but brings symmetry and serves to see the extension $A \wedge A \equiv A, A \vee A \equiv A$.
- Classical: Negation needs a proper treatment
- Additive cut (and in general in all branching rules)

$G_C^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$

Axiom

$$A, \Gamma \vdash \Delta, A$$

Cut

$$\bowtie \frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

Structural Rules

$$>_L \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad >_R \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

Left Rules

$$\supset_L \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}$$

$$\wedge_L \frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

$$\vee_L \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}$$

$$\neg_L \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}$$

$$\forall_L \frac{A[t/x], \Gamma \vdash \Delta}{\forall x.A, \Gamma \vdash \Delta}$$

$$\exists_L \frac{A[y/x], \Gamma \vdash \Delta}{\exists x.A, \Gamma \vdash \Delta}$$

Right Rules

$$\supset_R \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}$$

$$\wedge_R \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee_R \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\neg_R \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

$$\forall_R \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A}$$

$$\exists_R \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A}$$

where in $\forall_R$ and $\exists_L$ the variable y is not free in the conclusion

Fig. 8 *First Order Classical Sequent Calculus with Cut* $G_C^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$

$G_C^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$

Axiom

Multisets, L/R

$$A, \Gamma \vdash \Delta, A$$

Cut

$$\bowtie \frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

Structural Rules

Extending the intuitionistic

$$>_L \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad >_R \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

Left Rules

$$\supset_L \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}$$

$$\wedge_L \frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

$$\vee_L \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}$$

Right Rules

$$\supset_R \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}$$

$$\wedge_R \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee_R \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

Negation

$$\neg_L \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \qquad \neg_R \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

1st order

$$\forall_L \frac{A[t/x], \Gamma \vdash \Delta}{\forall x.A, \Gamma \vdash \Delta} \qquad \forall_R \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A}$$

$$\exists_L \frac{A[y/x], \Gamma \vdash \Delta}{\exists x.A, \Gamma \vdash \Delta} \qquad \exists_R \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A}$$

where in $\forall_R$ and $\exists_L$ the variable y is not free in the conclusion

Fig. 8 *First Order Classical Sequent Calculus with Cut* $G_C^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$

$G_C^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$

Axiom

$$A, \Gamma \vdash \Delta, A$$

Cut

$$\bowtie \; \frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

Additive context management

Structural Rules

$$>_L \; \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad >_R \; \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

Left Rules

$$\supset_L \; \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}$$

$$\wedge_L \; \frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

$$\vee_L \; \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}$$

$$\neg_L \; \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}$$

$$\forall_L \; \frac{A[t/x], \Gamma \vdash \Delta}{\forall x.A, \Gamma \vdash \Delta}$$

$$\exists_L \; \frac{A[y/x], \Gamma \vdash \Delta}{\exists x.A, \Gamma \vdash \Delta}$$

Right Rules

$$\supset_R \; \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}$$

$$\wedge_R \; \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee_R \; \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\neg_R \; \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

$$\forall_R \; \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A}$$

$$\exists_R \; \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A}$$

where in $\forall_R$ and $\exists_L$ the variable y is not free in the conclusion

Fig. 8 *First Order Classical Sequent Calculus with Cut* $G_C^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$

$G_C^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$

Axiom

$$A, \Gamma \vdash \Delta, A$$

Cut

$$\bowtie \frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

Other symmetries

Structural Rules

$$>_L \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad >_R \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

Left Rules

$$\supset_L \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}$$

$$\wedge_L \frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

$$\vee_L \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}$$

$$\neg_L \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}$$

$$\forall_L \frac{A[t/x], \Gamma \vdash \Delta}{\forall x.A, \Gamma \vdash \Delta}$$

$$\exists_L \frac{A[y/x], \Gamma \vdash \Delta}{\exists x.A, \Gamma \vdash \Delta}$$

Right Rules

$$\supset_R \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}$$

$$\wedge_R \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

Different from intuitionistic

$$\vee_R \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\neg_R \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

$$\forall_R \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A}$$

$$\exists_R \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A}$$

where in $\forall_R$ and $\exists_L$ the variable y is not free in the conclusion

Fig. 8 *First Order Classical Sequent Calculus with Cut* $G_C^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$

$LK^{\bowtie}$

Modify $G_C^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$ and accomodate more symmetries:

- Different Axiom (thin) and Cut (multiplicative), and different $\wedge_L$, $\vee_R$
- Introduce weakening rules
- Weakening and contraction rules induce idempotency on multisets: they behave as sets $A \wedge A \equiv A, A \vee A \equiv A$

Axiom

$$A \vdash A$$

Cut

$$\bowtie\ \frac{\Gamma \vdash \Delta, A \quad A, \Lambda \vdash \Theta}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

Structural Rules

$$>_{\mathsf{L}}\ \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad >_{\mathsf{R}}\ \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} \qquad <_{\mathsf{L}}\ \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad <_{\mathsf{R}}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}$$

Left Rules

$$\supset_{\mathsf{L}}\ \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}$$

$$\wedge_{\mathsf{LL}}\ \frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \wedge_{\mathsf{LR}}\ \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

$$\vee_{\mathsf{L}}\ \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}$$

$$\neg_{\mathsf{L}}\ \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}$$

$$\forall_{\mathsf{L}}\ \frac{A[t/x], \Gamma \vdash \Delta}{\forall x.A, \Gamma \vdash \Delta}$$

$$\exists_{\mathsf{L}}\ \frac{A[y/x], \Gamma \vdash \Delta}{\exists x.A, \Gamma \vdash \Delta}$$

Right Rules

$$\supset_{\mathsf{R}}\ \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}$$

$$\wedge_{\mathsf{R}}\ \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee_{\mathsf{RL}}\ \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \qquad \vee_{\mathsf{RR}}\ \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\neg_{\mathsf{R}}\ \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

$$\forall_{\mathsf{R}}\ \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A}$$

$$\exists_{\mathsf{R}}\ \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A}$$

where in $\forall_{\mathsf{R}}$ and $\exists_{\mathsf{L}}$ the variable y is not free in the conclusion

Fig. 9 *First Order Classical Sequent Calculus with Multiplicative Cut* LK

$LK^{\bowtie}$

Axiom

$$A \vdash A$$

Cut

$$\bowtie \frac{\Gamma \vdash \Delta, A \quad A, \Lambda \vdash \Theta}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

Structural Rules

$$>_{L} \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad >_{R} \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

weakening

$$<_{L} \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad <_{R} \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}$$

Left Rules

$$\supset_{L} \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}$$

$$\wedge_{LL} \frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \wedge_{LR} \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

$$\vee_{L} \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}$$

Right Rules

$$\supset_{R} \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}$$

$$\wedge_{R} \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee_{RL} \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \qquad \vee_{RR} \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}$$

Negation

$$\neg_{L} \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \qquad \neg_{R} \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

1st order

$$\forall_{L} \frac{A[t/x], \Gamma \vdash \Delta}{\forall x.A, \Gamma \vdash \Delta} \qquad \forall_{R} \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A}$$

$$\exists_{L} \frac{A[y/x], \Gamma \vdash \Delta}{\exists x.A, \Gamma \vdash \Delta} \qquad \exists_{R} \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A}$$

where in $\forall_R$ and $\exists_L$ the variable y is not free in the conclusion

Fig. 9 *First Order Classical Sequent Calculus with Multiplicative Cut* LK

$LK^{\bowtie}$

Axiom

$$A \vdash A$$

Cut

$$\bowtie \frac{\Gamma \vdash \Delta, A \quad A, \Lambda \vdash \Theta}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

Multiplicative cut rule

Structural Rules

$$>_{L} \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad >_{R} \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} \qquad <_{L} \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad <_{R} \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}$$

Left Rules

$$\supset_{L} \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}$$

$$\wedge_{LL} \frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \wedge_{LR} \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

$$\vee_{L} \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}$$

$$\neg_{L} \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}$$

$$\forall_{L} \frac{A[t/x], \Gamma \vdash \Delta}{\forall x.A, \Gamma \vdash \Delta}$$

$$\exists_{L} \frac{A[y/x], \Gamma \vdash \Delta}{\exists x.A, \Gamma \vdash \Delta}$$

Right Rules

$$\supset_{R} \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}$$

$$\wedge_{R} \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee_{RL} \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \qquad \vee_{RR} \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\neg_{R} \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

$$\forall_{R} \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A}$$

$$\exists_{R} \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A}$$

where in $\forall_{R}$ and $\exists_{L}$ the variable y is not free in the conclusion

Fig. 9 *First Order Classical Sequent Calculus with Multiplicative Cut* LK

$LK^{\bowtie}$

Axiom

$$A \vdash A$$

Cut

$$\bowtie\ \frac{\Gamma \vdash \Delta, A \quad A, \Lambda \vdash \Theta}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

Structural Rules

$$>_{\mathsf{L}}\ \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad >_{\mathsf{R}}\ \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} \qquad <_{\mathsf{L}}\ \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad <_{\mathsf{R}}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}$$

Left Rules

$$\supset_{\mathsf{L}}\ \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta}$$

$$\wedge_{\mathsf{LL}}\ \frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \wedge_{\mathsf{LR}}\ \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

$$\vee_{\mathsf{L}}\ \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta}$$

$$\neg_{\mathsf{L}}\ \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}$$

$$\forall_{\mathsf{L}}\ \frac{A[t/x], \Gamma \vdash \Delta}{\forall x.A, \Gamma \vdash \Delta}$$

$$\exists_{\mathsf{L}}\ \frac{A[y/x], \Gamma \vdash \Delta}{\exists x.A, \Gamma \vdash \Delta}$$

Right Rules

$$\supset_{\mathsf{R}}\ \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}$$

$$\wedge_{\mathsf{R}}\ \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee_{\mathsf{RL}}\ \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \qquad \vee_{\mathsf{RR}}\ \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}$$

same as in intuitionistic

$$\neg_{\mathsf{R}}\ \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$$

$$\forall_{\mathsf{R}}\ \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A}$$

$$\exists_{\mathsf{R}}\ \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A}$$

where in $\forall_{\mathsf{R}}$ and $\exists_{\mathsf{L}}$ the variable y is not free in the conclusion

Fig. 9 *First Order Classical Sequent Calculus with Multiplicative Cut* LK

Examples in $G_C^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$ (top) and $LK^{\bowtie}$ (bottom)

$$\vee_R \dfrac{\neg_R \dfrac{A \vdash A}{\vdash A, \neg A}}{\vdash A \vee \neg A}$$

$$\supset_R \dfrac{\supset_L \dfrac{\supset_R \dfrac{A \vdash B, A}{\vdash A \supset B, A} \quad A \vdash A}{(A \supset B) \supset A \vdash A}}{\vdash ((A \supset B) \supset A) \supset A}$$

$$>_R \dfrac{\vee_{RL} \dfrac{\vee_{RR} \dfrac{\neg_R \dfrac{A \vdash A}{\vdash A, \neg A}}{\vdash A, A \vee \neg A}}{\vdash A \vee \neg A, A \vee \neg A}}{\vdash A \vee \neg A}$$

$$\supset_R \dfrac{\supset_L \dfrac{\supset_R \dfrac{<_R \dfrac{A \vdash A}{A \vdash B, A}}{\vdash A \supset B, A} \quad A \vdash A}{(A \supset B) \supset A \vdash A}}{\vdash ((A \supset B) \supset A) \supset A}$$

LK: think ahead if contraction is needed. Contraction on any formula

Examples in *LK*, first order

Bottom-up, more than one rule is applicabile – how to choose?

- if more than two quantifier rules are applicable, try and apply first one with a proviso,
- and delay application of quantifier rules otherwise, if possible

$$\supset_R\,\dfrac{\forall_R\,\dfrac{\forall_L\,\dfrac{\supset_L\,\dfrac{<_R\,\dfrac{A\vdash A}{A\vdash A,B}\qquad <_L\,\dfrac{B\vdash B}{B,A\vdash B}}{A\supset B, A\vdash B}}{\forall x.(A\supset B), A\vdash B}}{\forall x.(A\supset B), A\vdash \forall x.B}}{\supset_R\,\dfrac{\forall x.(A\supset B)\vdash A\supset\forall x.B}{\vdash \forall x.(A\supset B)\supset(A\supset\forall x.B)}}$$

where x is not free in A.

$$\supset_R\,\dfrac{\supset_L\,\dfrac{\forall_R\,\dfrac{\exists_R\,\dfrac{\supset_R\,\dfrac{<_R\,\dfrac{A\vdash A}{A\vdash A,B}}{\vdash A, A\supset B}}{\vdash A,\exists x.(A\supset B)}}{\vdash \forall x.A,\exists x.(A\supset B)}\qquad \exists_R\,\dfrac{\supset_R\,\dfrac{<_L\,\dfrac{B\vdash B}{A,B\vdash B}}{B\vdash A\supset B}}{B\vdash\exists x.(A\supset B)}}{\forall x.A\supset B\vdash\exists x.(A\supset B)}}{\vdash(\forall x.A\supset B)\supset\exists x.(A\supset B)}$$

Equivalence of $G_c^{\supset,\wedge,\vee,\neg,\forall,\exists,\bowtie}$ and $LK^{\bowtie}$

- We need two effective procedures, to transform proofs in one system to equivalent proofs in the other one (by structural induction)
- E.g., base case and an inductive case to transform a proof Π in LK

$$\Pi = A, \Gamma \vdash \Delta, A \qquad f(\Pi) = \begin{array}{l} \phantom{<_L}\; A \vdash A \\ <_L \overline{} \\ \phantom{<_L}\quad \vdots \\ <_L \overline{} \\ \phantom{<_L}\; A, \Gamma \vdash A \\ <_R \overline{} \\ \phantom{<_R}\quad \vdots \\ <_R \overline{} \\ \phantom{<_R}\; A, \Gamma \vdash \Delta, A \end{array}$$

$$\Pi = \begin{array}{c} \Pi' \\ \wedge_L \dfrac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \end{array} \qquad f(\Pi) = \begin{array}{c} f(\Pi') \\ A, B, \Gamma \vdash \Delta \\ \wedge_{LL} \overline{} \\ A \wedge B, B, \Gamma \vdash \Delta \\ \wedge_{LR} \overline{} \\ A \wedge B, A \wedge B, \Gamma \vdash \Delta \\ >_L \overline{} \\ A \wedge B, \Gamma \vdash \Delta \end{array}$$

Equivalence of $G_C^{\supset,\wedge,\vee,\neg,\forall,\exists}$ and $LK^{\bowtie}$

- ..and conversely, the base case to transform a proof Π in LK

$$\Pi = A \vdash A \qquad\qquad g(\Pi) = A \vdash A$$

- .. the case of a proof whose lowermost rule is a cut, with auxiliary functions $\cdot \leftarrow \cdot$ and $\cdot \rightarrow \cdot$ to "pad" the whole subproofs (i.e. all its sequents, both on the left and the right) that may require some renaming

$$\Pi = \dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, A} \qquad \dfrac{\Pi_2}{A, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}\bowtie$$

$$g(\Pi) = \dfrac{\dfrac{\Lambda \rightarrow g(\Pi_1) \leftarrow \Theta}{\Gamma, \Lambda \vdash \Delta, \Theta, A} \qquad \dfrac{\Gamma \rightarrow g(\Pi_2) \leftarrow \Delta}{A, \Gamma, \Lambda \vdash \Delta, \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}\bowtie$$

About Cut-Elimination

- No cut-elimination means no proof theory – compositionality
- $G_C^{\supset,\wedge,\vee,\neg,\forall,\exists}$ and $LK^{\bowtie}$ satisfy cut-elimination: a proof with cut is transformed into a cut-free one;
- a proof with cut is good for analysis, but a cut-free proof is related to proof search: both proofs are needed for different purposes;
- Refer to the literature for full details on the proof of cut elimination, we will just sketch, later on, the general idea only.
- Examples of applications of cut-elimination:
  - consistency (useful when no semantics is available), interpolation, Herbrand theorem, separation property, preservation of sign in a derivation....
  - for proof search (reduced search space), automated deduction (oracles and interactive theorem proving),..
- Boundaries: What is the cut formula – a generic formula? An atomic formula? (infinite choices in both cases)

About Cut-Elimination

- Syntactical proofs of c.e.: techniques based on permutability of rules or Girard-Tait
  - Provides an algorithm, and it is a "global" transformation on the whole proof;
  - the cost of the transformation is hyperexponential, sometimes more efficient than semantical means;
  - the transformed cut-free proof is usually not smaller in size;
  - the only possibility if semantics is not known or is not straightforward
- Semantical proofs of c.e. (test the truth of a sequent $\Gamma \vdash \Delta$: try to give an evaluation that satisfies Γ and falsifies Δ; a sequent is formally derivable by the rules if there is no refuting evaluation)
  - not constructive (i.e. no algorithm)
  - implies a completeness theorem

Example proof after cut-elimination

Proof with cut, and after cut-elimination

$$
\mathsf{c_R}\,\dfrac{\mathsf{cut}\,\dfrac{
\mathsf{\forall_R}\,\dfrac{\mathsf{\exists_R}\,\dfrac{\mathsf{\forall_R}\,\dfrac{\mathsf{\supset_R}\,\dfrac{\mathsf{w_R}\,\dfrac{p(z) \vdash p(z)}{p(z) \vdash p(z), p(y)}}{\vdash p(z), p(z) \supset p(y)}}{\vdash p(z), \forall y.(p(z) \supset p(y))}}{\vdash p(z), \exists x.\forall y.(p(x) \supset p(y))}}{\vdash \forall z.p(z), \exists x.\forall y.(p(x) \supset p(y))}
\qquad
\mathsf{\exists_R}\,\dfrac{\mathsf{\forall_R}\,\dfrac{\mathsf{\forall_L}\,\dfrac{\mathsf{\supset_R}\,\dfrac{\mathsf{w_L}\,\dfrac{p(y) \vdash p(y)}{p(x), p(y) \vdash p(y)}}{p(y) \vdash p(x) \supset p(y)}}{\forall z.p(z) \vdash p(x) \supset p(y)}}{\forall z.p(z) \vdash \forall y.(p(x) \supset p(y))}}{\forall z.p(z) \vdash \exists x.\forall y.(p(x) \supset p(y))}
}{\vdash \exists x.\forall y.(p(x) \supset p(y)), \exists x.\forall y.(p(x) \supset p(y))}}{\vdash \exists x.\forall y.(p(x) \supset p(y))}
$$

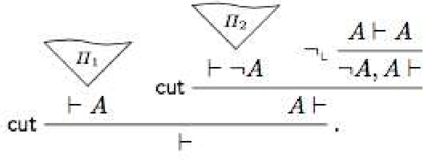
cut formula

$$
\mathsf{c_R}\,\dfrac{\mathsf{\exists_R}\,\dfrac{\mathsf{\forall_R}\,\dfrac{\mathsf{\exists_R}\,\dfrac{\mathsf{\forall_R}\,\dfrac{\mathsf{\supset_R}\,\dfrac{\mathsf{w_R}\,\dfrac{\mathsf{\supset_R}\,\dfrac{\mathsf{w_L}\,\dfrac{p(y) \vdash p(y)}{p(y), p(x) \vdash p(y)}}{p(y) \vdash p(x) \supset p(y)}}{p(y) \vdash p(z), p(x) \supset p(y)}}{\vdash p(y) \supset p(z), p(x) \supset p(y)}}{\vdash \forall z.(p(y) \supset p(z)), p(x) \supset p(y)}}{\vdash \exists x.\forall y.(p(x) \supset p(y)), p(x) \supset p(y)}}{\vdash \exists x.\forall y.(p(x) \supset p(y)), \forall y.(p(x) \supset p(y))}}{\vdash \exists x.\forall y.(p(x) \supset p(y)), \exists x.\forall y.(p(x) \supset p(y))}}{\vdash \exists x.\forall y.(p(x) \supset p(y))}\;.
$$

Example: Consistency via Cut-Elimination

Prove that *LK* is consistent.

- Suppose *LK* inconsistent: then there are proofs for $\vdash A$ and $\vdash \neg A$;
- they can be composed with cut as follows, proving $\vdash$:



- Cut elimination holds in *LK*, so there exists a cut-free proof for $\vdash$;
- $\vdash$ is neither an axiom nor a conclusion of any rule: contradiction.

Example: Symmetries and Atomicity

The axiom of LK can be replaced by one in atomic form, $a \vdash a$

- Induction on the structure of A: the sequent of the form $A \vdash A$ is replaced by a proof

$$\begin{array}{c} \Pi_A \\ A \vdash A \end{array}$$

- E.g. two inductive cases are

$$\wedge_R \dfrac{\wedge_{LL}\dfrac{\begin{array}{c}\Pi_B \\ B \vdash B\end{array}}{B \wedge C \vdash B} \quad \wedge_{LR}\dfrac{\begin{array}{c}\Pi_C \\ C \vdash C\end{array}}{B \wedge C \vdash C}}{B \wedge C \vdash B \wedge C}, \qquad \vee_L \dfrac{\vee_{RL}\dfrac{\begin{array}{c}\Pi_B \\ B \vdash B\end{array}}{B \vdash B \vee C} \quad \vee_{RR}\dfrac{\begin{array}{c}\Pi_C \\ C \vdash C\end{array}}{C \vdash B \vee C}}{B \vee C \vdash B \vee C},$$

  and the other cases are similar.
- "Verifying" an axiom requires constant time.
- Similar result for weakening, but not for contraction.
- Reducing cut to atomic form would require to re-do the cut-elimination theorem from scratch.

LK - Some Interim Activities

- Consider a proof system LK^* obtained by modifying LK this way:
  - In its sequents, Γ and Δ are sets rather than multisets;
  - it does not have any contraction nor weakening rules;
  - there is no cut rule.

  Can you formally prove that LK^* and LK are equivalent proof system (provability is preserved)? Or can you find a counterexample?
- Complete the proof to show that the cut rule in LK can be replaced by a cut rule on atomic formulae.

Proof of Cut Elimination - Intuition

- Study of permutability of instances of cut rules in a proof, bottom-up, inductively.
- Requires a (complex) induction measure to guarantee termination of the procedure. The measure takes into account
  - how deep in the proof the instance of cut rule is,
  - how complex the cut formula is,
  - the immediate subproofs/surroundings of the instance of cut
- An easy case, when things work fine.. cut formula is $B \vee C$

$$\bowtie\dfrac{\vee_R\dfrac{\overset{\Pi_1}{\Gamma \vdash \Delta, B}}{\Gamma \vdash \Delta, B \vee C} \qquad \vee_L\dfrac{\overset{\Pi_2}{B, \Lambda \vdash \Theta} \quad \overset{\Pi_3}{C, \Lambda \vdash \Theta}}{B \vee C, \Lambda \vdash \Delta}}{\Gamma, \Lambda \vdash \Delta, \Theta} \quad \Longrightarrow \quad \bowtie\dfrac{\overset{\Pi_1}{\Gamma \vdash \Delta, B} \quad \overset{\Pi_2}{B, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

- Two nasty cases in the proof will eventually suggest not to work with the cut rule itself for the proof of cut elimination, rather a generalisation of the rule.

Proof of Cut Elimination - Intuition

- Nasty one: there is a contraction just above the cut

$$\bowtie\dfrac{>_R\dfrac{\overset{\Pi_1}{\Gamma \vdash \Delta, A, A}}{\Gamma \vdash \Delta, A} \qquad \overset{\Pi_2}{A, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta} \quad \Longrightarrow^{?} \quad >_L;>_R\dfrac{\bowtie\dfrac{\bowtie\dfrac{\overset{\Pi_1}{\Gamma \vdash \Delta, A, A} \quad \overset{\Pi_2}{A, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, A} \qquad \overset{\Pi_2}{A, \Lambda \vdash \Theta}}{\Gamma, \Lambda, \Lambda \vdash \Delta, \Theta, \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}\,\star$$

- .. one extra cut, a much bigger proof, requiring duplication of contexts..

Proof of Cut Elimination - Intuition

- Nasty two: two contractions just above the cut, repeating the Left/Right pattern

$$\bowtie\dfrac{>_R\dfrac{\overset{\Pi_1}{\Gamma \vdash \Delta, C, C}}{\Gamma \vdash \Delta, C} \qquad >_L\dfrac{\overset{\Pi_2}{C, C, \Lambda \vdash \Theta}}{C, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta} \quad \Longrightarrow^{?} \quad >_R>_L\dfrac{\bowtie\dfrac{\bowtie\dfrac{\overset{\Pi_1}{\Gamma \vdash \Delta, C, C} \qquad >_L\dfrac{\overset{\Pi_2}{C, C, \Lambda \vdash \Theta}}{C, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta, C} \qquad >_L\dfrac{\overset{\Pi_2}{C, C, \Lambda \vdash \Theta}}{C, \Lambda \vdash \Theta}}{\Gamma, \Lambda, \Lambda \vdash \Delta, \Theta, \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}\star$$

- .. the transformed proof would present again a case on contraction just above a cut (dual of the previous nasty case). No termination.

Proof of Cut Elimination - Intuition

- Solution to handle the two nasty cases: Use a generalised cut rule for the cut elimination proof

$$\bowtie^{+}\ \frac{\Gamma \vdash \Delta, mA \quad nA, \Lambda \vdash \Theta}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

$$\bowtie^{+}\ \dfrac{>_{L}\ \dfrac{\overset{\Pi_1}{\Gamma \vdash \Delta, (m-1)A, A, A}}{\Gamma \vdash \Delta, mA} \quad \overset{\Pi_2}{nA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta} \quad \Longrightarrow^{?} \quad \bowtie^{+}\ \dfrac{\overset{\Pi_1}{\Gamma \vdash \Delta, (m+1)A} \quad \overset{\Pi_2}{nA, \Lambda \vdash \Theta}}{\Gamma, \Lambda \vdash \Delta, \Theta}$$

- Γ, nA multiset with $m + n$ occurrences of A, when Γ contains m occurrences of A ($m, n > 0$).
- When $m = n = 1$ the generalised cut rule coincides with the usual one (simulated by cut and contractions).
- The cut-elimination proof will then terminate.

Variant of LK – One Sided GS1p

- Only right side of sequents (multisets): halves the number of rules
- Negation pushed to the atoms, formulae in n.n.f ($\bar{A}$ rather than $\neg A$)
- Atomic axiom admissible (as in *LK*).

$$\text{Ax}\ \frac{}{\vdash A, \bar{A}} \qquad \text{Cut}\ \frac{\vdash \Phi, A \quad \vdash \Psi, \bar{A}}{\vdash \Phi, \Psi}$$

$$\text{R}\vee_L\ \frac{\vdash \Phi, A}{\vdash \Phi, A \vee B} \qquad \text{R}\vee_R\ \frac{\vdash \Phi, B}{\vdash \Phi, A \vee B} \qquad \text{R}\wedge\ \frac{\vdash \Phi, A \quad \vdash \Phi, B}{\vdash \Phi, A \wedge B}$$

$$\text{RC}\ \frac{\vdash \Phi, A, A}{\vdash \Phi, A} \qquad \text{RW}\ \frac{\vdash \Phi}{\vdash \Phi, A}$$

Variant – Invertible Rules and G3

Invertible Rule: when from the derivability of its conclusion, the derivability of its premiss(es) follows.

- E.g. $\wedge_L$ is invertible in G_C but not in LK:

$$\wedge_L \frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \wedge_{LL} \frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad \wedge_{LR} \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta}$$

- A feature that can be useful for bottom-up proof search;
- G3-style systems (Ketonen '44)

$$\text{Ax} \frac{}{a, \Gamma \vdash \Delta, a} \qquad \text{cut} \frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

$$\bot_L \frac{}{\bot, \Gamma \vdash \Delta}$$

$$\wedge_L \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge_R \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee_L \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad \vee_R \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\supset_L \frac{\Gamma \vdash A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta} \qquad \supset_R \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}$$

Example $G3c$ (classical)

- Variant of G_C, negation defined as $\neg A = A \supset \bot$;
- atomic check on axiom, but in contexts Γ and Δ
- additive context in logical rules; multiplicative cut
- cut-elimination holds
- invertible rules make weakening and contraction admissible

$$\text{Ax}\ \frac{}{a, \Gamma \vdash \Delta, a} \qquad \text{cut}\ \frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

$$\bot_L\ \frac{}{\bot, \Gamma \vdash \Delta}$$

$$\wedge_L\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge_R\ \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee_L\ \frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad \vee_R\ \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}$$

$$\supset_L\ \frac{\Gamma \vdash A \quad B, \Gamma \vdash \Delta}{A \supset B, \Gamma \vdash \Delta} \qquad \supset_R\ \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \supset B}$$

$$\supset_R\ \dfrac{\supset_L\ \dfrac{\vee_R\ \dfrac{\supset_R\ \dfrac{Ax\ \dfrac{}{A \vdash A}}{\vdash A, \neg A}}{\vdash A \vee \neg A} \quad \bot_L\ \dfrac{}{\bot \vdash \bot}}{\neg(A \vee \neg A) \vdash \bot}}{\vdash \neg\neg(A \vee \neg A)}$$

Conclusions and Notes

- Gallier's coursenotes [2] available on the web (in revised version) is the most suitable reference, for the level of detail, for many topics presented today.
- Parts on the $G3_C$ systems may be found in traditional books, such as Troelstra and Schwichtenberg's book [3]. A quick tutorial on proof theory addressing proof search is this [1]. Both references are anyway departing very quickly from the intended focus and scope of this specific course.
- These topics are all quite standard to be widely offered, for example at ESSLLI. Old courses of mine at TU Dresden offer direct links to some resources, e.g. `http://www.cs.bath.ac.uk/pb/EMCL/2012/SPAL-12/index.html`
- Some of the interim activities proposed would require more time and are there just for the interested reader.

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[1] Paola Bruscoli & Alessio Guglielmi (2003): *A Tutorial on Proof Theoretic Foundations of Logic Programming.*
In Catuscia Palamidessi, editor: *Logic Programming, 19th International Conference (ICLP)*, *Lecture Notes in Computer Science* 2916, Springer-Verlag, pp. 109–127.
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[2] Jean Gallier (1993): *Constructive Logics. Part I: A Tutorial on Proof Systems and Typed λ-Calculi.*
Theoretical Computer Science 110, pp. 249–339.

[3] A.S. Troelstra & H. Schwichtenberg (1996): *Basic Proof Theory.*
Cambridge Tracts in Theoretical Computer Science 43, Cambridge University Press.