Incompleteness for higher order arithmetic and the limit of incompleteness

Yong Cheng

School of Philosophy
Wuhan University, Wuhan, China
International workshop on Proof Theory 2018, Ghent, Belgium

Modern logic and Philosophy

Modern logic updates and deepens our understanding of the following core concepts of philosophy:

Modern logic and Philosophy

Modern logic updates and deepens our understanding of the following core concepts of philosophy:

- Truth, Proof, Infinity, Computation, Set, Number, Definability

Modern logic and Philosophy

Modern logic updates and deepens our understanding of the following core concepts of philosophy:

- Truth, Proof, Infinity, Computation, Set, Number, Definability
- Self-reference, *Incompleteness*, Independence, Decidability

Modern logic and Philosophy

Modern logic updates and deepens our understanding of the following core concepts of philosophy:

- Truth, Proof, Infinity, Computation, Set, Number, Definability
- Self-reference, *Incompleteness*, Independence, Decidability
- Implication, Consistency, Paradox, Contradiction

Modern logic and Philosophy

Modern logic updates and deepens our understanding of the following core concepts of philosophy:

- Truth, Proof, Infinity, Computation, Set, Number, Definability
- Self-reference, *Incompleteness*, Independence, Decidability
- Implication, Consistency, Paradox, Contradiction
- Absoluteness, Knowability, Necessity, Vagueness, etc.

Part One: The current state

Outline

The current state of research:

Outline

The current state of research:

(1) Property of provability and truth

Outline

The current state of research:

(1) Property of provability and truth

(2) Generalization of Incompleteness theorem to arithmetical definable theory

Outline

The current state of research:

(1) Property of provability and truth

(2) Generalization of Incompleteness theorem to arithmetical definable theory

(3) Self-reference-free proof of incompleteness theorem

Outline

The current state of research:

(1) Property of provability and truth

(2) Generalization of Incompleteness theorem to arithmetical definable theory

(3) Self-reference-free proof of incompleteness theorem

(4) Proving Incompleteness via logical paradox

Outline

The current state of research:

(1) Property of provability and truth

(2) Generalization of Incompleteness theorem to arithmetical definable theory

(3) Self-reference-free proof of incompleteness theorem

(4) Proving Incompleteness via logical paradox

(5) The intensionality of **G2** for **PA**

Outline

The current state of research:

(1) Property of provability and truth
(2) Generalization of Incompleteness theorem to arithmetical definable theory
(3) Self-reference-free proof of incompleteness theorem
(4) Proving Incompleteness via logical paradox
(5) The intensionality of **G2** for **PA**
(6) Incompleteness and provability logic

Gödel's incompleteness theorem

Two goals of Hilbert's program:

Completeness A proof that all true mathematical statements can be proved in the formalism of mathematics.

Consistency A proof that no contradiction can be obtained in the formalism of mathematics using only "finitistic" reasoning about finite mathematical objects.

Gödel's incompleteness theorem

Two goals of Hilbert's program:

Completeness A proof that all true mathematical statements can be proved in the formalism of mathematics.

Consistency A proof that no contradiction can be obtained in the formalism of mathematics using only "finitistic" reasoning about finite mathematical objects.

Theorem (Gödel-Rosser)

(1) *Gödel-Rosser first incompleteness theorem* (**G1**)*: If T is a recursively axiomatized consistent extension of* **PA***, then T is not complete.*

(2) *Gödel's second incompleteness theorem* (**G2**)*: If T is a recursively axiomatized consistent extension of* **PA***, then the consistency of T is not provable in T.*

Provability and Truth

Definition

1. **Prof**$=\{\ulcorner\phi\urcorner : \phi$ *is sentence and* **PA** $\vdash \phi\}$.
2. **Truth**$=\{\ulcorner\phi\urcorner : \phi$ *is sentence and* $\mathfrak{N} \models \phi\}$ *where* $\mathfrak{N} = (\mathbb{N}, +, \cdot)$.

Theorem (Tarski's theorem on undefinability of truth)

Truth *is not definable in* $\mathfrak{N}$.

Provability and Truth

Definition

1. **Prof**$=\{\ulcorner\phi\urcorner : \phi$ *is sentence and* **PA** $\vdash \phi\}$.
2. **Truth**$=\{\ulcorner\phi\urcorner : \phi$ *is sentence and* $\mathfrak{N} \models \phi\}$ *where* $\mathfrak{N} = (\mathbb{N}, +, \cdot)$.

Theorem (Tarski's theorem on undefinability of truth)

Truth *is not definable in* $\mathfrak{N}$.

Truth	**Prof**
not definable in $\mathfrak{N}$	definable in $\mathfrak{N}$
not arithmetic	recursive eumerable
not recursive	not recursive
not representable in **PA**	not representable in **PA**
productive	not productive

Solovay's arithmetical completeness theorem

Definition

An arithmetic interpretation is a function that assigns to each formula of modal logic a sentence of the language of arithmetic.

Theorem (Solovay)

Arithmetical completeness theorem for GL *For any modal formula* ϕ, $\mathbf{GL} \vdash \phi$ *iff for every arithmetic interpretation* f, $\mathbf{PA} \vdash \phi^f$.

Arithmetical completeness theorem for GLS *For any modal formula* ϕ, $\mathbf{GLS} \vdash \phi$ *iff for every arithmetic interpretation* f, $\mathfrak{N} \models \phi^f$.

Definition

(1) *We say T is Σ_n-definable iff there is a Σ_n formula $\alpha(x)$ such that $\{n \in \omega : \mathfrak{N} \models \alpha(\overline{n})\} = \{\ulcorner\phi\urcorner : \phi \in T\}$.*

(2) *We say T is Σ_n-sound if and only if for all Σ_n sentences ϕ, if $T \vdash \phi$, then $\mathfrak{N} \models \phi$.*

- Gödel's incompleteness theorem hold for Σ_1-definable theories containing **PA**.
- We generalize Gödel's incompleteness theorem for arithmetically definable theories.

Definition

(1) *We say* T *is* Σ_n*-definable iff there is a* Σ_n *formula* $\alpha(x)$ *such that* $\{n \in \omega : \mathfrak{N} \models \alpha(\overline{n})\} = \{\ulcorner \phi \urcorner : \phi \in T\}$.

(2) *We say* T *is* Σ_n*-sound if and only if for all* Σ_n *sentences* ϕ*, if* $T \vdash \phi$*, then* $\mathfrak{N} \models \phi$.

- Gödel's incompleteness theorem hold for Σ_1-definable theories containing **PA**.
- We generalize Gödel's incompleteness theorem for arithmetically definable theories.

Theorem (Kikuchi, Kurahashi, 2017)

(1) *Every* Σ_{n+1}*-definable* Σ_n*-sound theory is incomplete.*

(2) *Every consistent theory having* Π_{n+1} *set of theorems has a true but unprovable* Π_n *sentence.*

(3) *Any* Σ_{n+1}*-definable* Σ_n*-sound theory can not prove its own* Σ_n*-soundness.*

Different proofs of incompleteness theorem

- Constructive proof: directly construct the independence sentence
- Proof via diagonalization lemma
- Proof via logical paradox
- Proof via recursion theory
- Proof via model theory

Different proofs of incompleteness theorem

- Constructive proof: directly construct the independence sentence
- Proof via diagonalization lemma
- Proof via logical paradox
- Proof via recursion theory
- Proof via model theory

Question

Could we give a self-reference-free proof of Gödel's incompleteness theorem?

Incompleteness theorem and logical paradox

- Incompleteness is closely related to paradox.
- "Any epistemological antinomy could be used for a similar proof of the existence of undecidable propositions"—-Gödel

Incompleteness theorem and logical paradox

- Incompleteness is closely related to paradox.
- "Any epistemological antinomy could be used for a similar proof of the existence of undecidable propositions"—-Gödel

Different proofs of incompleteness theorem via paradox:

Gödel	Liar Paradox
Boolos	Berry's paradox
Kurahashi	Yablo's Paradox
Kritchman	Unexpected Examination Paradox
Cieśliński	Grelling's paradox

Numeration and provability predicate

Definition

Let T be any recursively axiomatized consistent extension of **PA** *and $\alpha(x)$ be a formula in the same language.*

1. *$\alpha(x)$ is a numeration of T if for any $n, \mathbf{PA} \vdash \alpha(\overline{n})$ iff n is the Gödel number of some axiom of T.*
2. *Let $\alpha(x)$ be a numeration of T. Define the formula $\mathbf{Prf}_\alpha(x, y)$ saying "y is the Gödel number of a proof of the formula with Gödel number x from the set of all sentences satisfying $\alpha(x)$".*
3. *Define the provability predicate $\mathbf{Pr}_\alpha(x)$ of $\alpha(x)$ as $\mathbf{Pr}_\alpha(x) \triangleq \exists y \mathbf{Prf}_\alpha(x, y)$ and consistency statement $\mathbf{Con}_\alpha$ as $\triangleq \neg \mathbf{Pr}_\alpha(\perp)$.*

Drivability Conditions and **G2**

Let T be a recursively axiomatized consistent extension of **PA** and $\alpha(x)$ be any Σ_1 numeration of T. Then $\mathbf{Pr}_\alpha(x)$ satisfies the following properties:

D1 If $T \vdash \varphi$, then $\mathbf{PA} \vdash \mathbf{Pr}_\alpha(\overline{\ulcorner\varphi\urcorner})$;

D2 If φ is Σ_1 sentence, then $\mathbf{PA} \vdash \varphi \rightarrow \mathbf{Pr}_\alpha(\overline{\ulcorner\varphi\urcorner})$;

D3 $\mathbf{PA} \vdash \mathbf{Pr}_\alpha(\overline{\ulcorner\varphi\urcorner}) \rightarrow (\mathbf{Pr}_\alpha(\overline{\ulcorner\varphi \rightarrow \psi\urcorner}) \rightarrow \mathbf{Pr}_\alpha(\overline{\ulcorner\psi\urcorner}))$.

Drivability Conditions and **G2**

Let T be a recursively axiomatized consistent extension of **PA** and $\alpha(x)$ be any Σ_1 numeration of T. Then $\mathbf{Pr}_\alpha(x)$ satisfies the following properties:

- **D1** If $T \vdash \varphi$, then $\mathbf{PA} \vdash \mathbf{Pr}_\alpha(\overline{\ulcorner \varphi \urcorner})$;
- **D2** If φ is Σ_1 sentence, then $\mathbf{PA} \vdash \varphi \to \mathbf{Pr}_\alpha(\overline{\ulcorner \varphi \urcorner})$;
- **D3** $\mathbf{PA} \vdash \mathbf{Pr}_\alpha(\overline{\ulcorner \varphi \urcorner}) \to (\mathbf{Pr}_\alpha(\overline{\ulcorner \varphi \to \psi \urcorner}) \to \mathbf{Pr}_\alpha(\overline{\ulcorner \psi \urcorner}))$.

Theorem (**G2**, Gödel)

*Let T be any recursively axiomatized consistent extension of **PA**. If $\alpha(x)$ is any Σ_1 numeration of T, then $T \nvdash \mathbf{Con}_\alpha$.*

The intensionality of **G2** for **PA**

The intensional problem of **G2** Whether **G2** holds for **PA** depends on the numeration of **PA**.

Theorem (Feferman)

There exists a Π_1 *numeration* $\pi(x)$ *of* **PA** *such that* **G2** *fails:* **PA** $\vdash$ $\mathbf{Con}_{\pi}$.

The intensionality of **G2** for **PA**

The intensional problem of **G2** Whether **G2** holds for **PA** depends on the numeration of **PA**.

Theorem (Feferman)

There exists a Π_1 *numeration* $\pi(x)$ *of* **PA** *such that* **G2** *fails:* $\mathbf{PA} \vdash \mathbf{Con}_\pi$.

- Whether **G2** holds for **PA** depends on the numeration of **PA**.
- **D1-D3** are the sufficient condition but not the necessary condition to show that **G2** holds for **PA**.
- There exists a Σ_2 numeration $\alpha(x)$ of **PA** such that **D2** does not hold for $\mathbf{Pr}_\alpha(x)$ but **G2** holds for **PA**.

Incompleteness and provability logic

Let T be any recursively axiomatized consistent extension of **PA** and $\alpha(x)$ be a numeration of T. The provability logic $\mathbf{PL}_\alpha(T)$ is the set of all modal principles which are verifiable in T when the modal operator $\Box$ is interpreted as $\mathbf{Pr}_\alpha(x)$.

Incompleteness and provability logic

Let T be any recursively axiomatized consistent extension of **PA** and $\alpha(x)$ be a numeration of T. The provability logic $\mathbf{PL}_\alpha(T)$ is the set of all modal principles which are verifiable in T when the modal operator $\Box$ is interpreted as $\mathbf{Pr}_\alpha(x)$.

Theorem (Solovay's arithmetical completeness theorem)

*Let T be any recursively axiomatized consistent extension of **PA**. If T is Σ_1-sound, then for any Σ_1 numeration $\alpha(x)$ of T, the provability logic $\mathbf{PL}_\alpha(T)$ is precisely **GL**.*

Classification of provability logic under numeration

- The provability logic $\mathbf{PL}_\tau(T)$ of a Σ_n numeration $\tau(x)$ of T is a normal modal logic.
- We could classify provability logic according to the numeration of T.

Classification of provability logic under numeration

- The provability logic $\mathbf{PL}_\tau(T)$ of a Σ_n numeration $\tau(x)$ of T is a normal modal logic.
- We could classify provability logic according to the numeration of T.

Question

Which normal modal logic is a provability logic $\mathbf{PL}_\tau(T)$ of some Σ_n numeration $\tau(x)$ of T?

Classification of provability logic under numeration

- The provability logic $\mathbf{PL}_\tau(T)$ of a Σ_n numeration $\tau(x)$ of T is a normal modal logic.
- We could classify provability logic according to the numeration of T.

Question

Which normal modal logic is a provability logic $\mathbf{PL}_\tau(T)$ of some Σ_n numeration $\tau(x)$ of T?

Theorem (Kurahashi, 2018)

1. *For any recursively axiomatized consistent extension T of* **PA**, *there exists a Σ_2 numeration $\alpha(x)$ of T such that the provability logic $\mathbf{PL}_\alpha(T)$ is* **K**.
2. *For each $n \geq 2$, there exists a Σ_2 numeration $\tau(x)$ of T such that the provability logic $\mathbf{PL}_\tau(T)$ coincides with modal logic $\mathbf{K} + \Box(\Box^n p \to p) \to \Box p$.*

Part Two: Understanding incompleteness

Research Motivation

Motivation Understanding incompleteness: Exploring the relationship between incompleteness, self-reference, provability logic, logical paradox and formal theory of truth

In this talk, I focus on the following two questions about incompleteness:

Research Motivation

Motivation Understanding incompleteness: Exploring the relationship between incompleteness, self-reference, provability logic, logical paradox and formal theory of truth

In this talk, I focus on the following two questions about incompleteness:

1. Incompleteness for high order arithmetic

Research Motivation

Motivation Understanding incompleteness: Exploring the relationship between incompleteness, self-reference, provability logic, logical paradox and formal theory of truth

In this talk, I focus on the following two questions about incompleteness:

1. Incompleteness for high order arithmetic
2. The limit of Incompleteness for subsystems of **PA**

Mathematical examples of **G1** for **PA**

Gödel's proof of **G1** uses meta-mathematics and the independent sentence Gödel constructed (Gödel's sentence) is of meta-mathematical nature and has no real mathematical content.

Mathematical examples of **G1** for **PA**

Gödel's proof of **G1** uses meta-mathematics and the independent sentence Gödel constructed (Gödel's sentence) is of meta-mathematical nature and has no real mathematical content.

Question

Could we find a sentence about arithmetic with interesting mathematical contents which is independent of **PA***?*

Mathematical examples of **G1** for **PA**

Gödel's proof of **G1** uses meta-mathematics and the independent sentence Gödel constructed (Gödel's sentence) is of meta-mathematical nature and has no real mathematical content.

Question

Could we find a sentence about arithmetic with interesting mathematical contents which is independent of **PA***?*

Theorem (Paris-Harrington)

If **PA** *is consistent, then there exists a sentence ϕ of combinatorial contents such that $\mathfrak{N} \models \phi$, but ϕ is independent of* **PA***.*

Incompleteness for high order arithmetic

Definition

Definition of higher order arithmetic:

(1) $Z_2 = ZFC^- +$ *Every set is countable.*[1]

(2) $Z_3 = ZFC^- + \mathcal{P}(\omega)$ *exists* $+$ *Every set is of cardinality* $\leq \beth_1$.

(3) $Z_4 = ZFC^- + \mathcal{P}(\mathcal{P}(\omega))$ *exists* $+$ *Every set is of cardinality* $\leq \beth_2$.

Corollary

If Z_2 *is consistent, then there is a true sentence about analysis which is not provable in* Z_2.

[1] ZFC^- denotes ZFC with the Power Set Axiom deleted and Collection instead of Replacement.

Fact

Many classic mathematical theorems about analysis which are expressible in Z_2 *are provable in* Z_2*.*

Fact

Many classic mathematical theorems about analysis which are expressible in Z_2 *are provable in* Z_2*.*

Question

Relativized Hilbert's program to Z_2 *Is* Z_2 *complete for classic mathematical theorems expressible in* Z_2*?*

Fact

Many classic mathematical theorems about analysis which are expressible in Z_2 are provable in Z_2.

Question

Relativized Hilbert's program to Z_2 *Is Z_2 complete for classic mathematical theorems expressible in Z_2?*

Motivation Finding a counterexample for this question which is expressible in Z_2 but not provable in Z_2.

Harrington's Theorem

Harrington's theorem $Det(\Sigma^1_1)$ implies $0^\sharp$ exists.

Definition

We let Harrington's Principle, HP *for short, denote the following statement:* $\exists x \in 2^\omega \forall \alpha(\alpha$ *is countable* x*-admissible* $\rightarrow \alpha$ *is an L-cardinal).*

Harrington's Theorem

Harrington's theorem $Det(\Sigma_1^1)$ implies $0^\sharp$ exists.

Definition

We let Harrington's Principle, HP *for short, denote the following statement:* $\exists x \in 2^\omega \forall \alpha(\alpha$ *is countable* x*-admissible* $\rightarrow \alpha$ *is an L-cardinal).*

Harrington's proof of "$Det(\Sigma_1^1)$ implies $0^\sharp$ exists" in ZF is done in two steps:

Harrington's Theorem

Harrington's theorem $Det(\Sigma^1_1)$ implies $0^\sharp$ exists.

Definition

We let Harrington's Principle, HP *for short, denote the following statement:* $\exists x \in 2^\omega \forall \alpha(\alpha$ *is countable* x*-admissible* $\rightarrow \alpha$ *is an L-cardinal).*

Harrington's proof of "$Det(\Sigma^1_1)$ implies $0^\sharp$ exists" in ZF is done in two steps:

First Step $Det(\Sigma^1_1)$ implies HP;

Harrington's Theorem

Harrington's theorem $Det(\Sigma^1_1)$ implies $0^\sharp$ exists.

Definition

We let Harrington's Principle, HP *for short, denote the following statement:* $\exists x \in 2^\omega \forall \alpha(\alpha$ *is countable* x*-admissible* $\rightarrow \alpha$ *is an L-cardinal).*

Harrington's proof of "$Det(\Sigma^1_1)$ implies $0^\sharp$ exists" in ZF is done in two steps:

First Step $Det(\Sigma^1_1)$ implies HP;

Second Step HP implies $0^\sharp$ exists.

Harrington's Theorem

Harrington's theorem $Det(\Sigma^1_1)$ implies $0^\sharp$ exists.

Definition

We let Harrington's Principle, HP *for short, denote the following statement:* $\exists x \in 2^\omega \forall \alpha(\alpha$ *is countable* x*-admissible* $\rightarrow \alpha$ *is an* L*-cardinal).*

Harrington's proof of "$Det(\Sigma^1_1)$ implies $0^\sharp$ exists" in ZF is done in two steps:

First Step $Det(\Sigma^1_1)$ implies HP;

Second Step HP implies $0^\sharp$ exists.

In ZF we have

$$Det(\Sigma^1_1) \Leftrightarrow \text{HP} \Leftrightarrow 0^\sharp \text{ exists.}$$

The counterexample

- The first step "$Det(\Sigma^1_1)$ implies HP" is provable in Z_2.

The counterexample

- The first step "$Det(\Sigma^1_1)$ implies HP" is provable in Z_2.

Question

Is "HP implies $0^\sharp$ exists" provable in Z_2?

The counterexample

- The first step "$Det(\Sigma^1_1)$ implies HP" is provable in Z_2.

Question

Is "HP implies $0^\sharp$ exists" provable in Z_2?

The counterexample I find is the sentence: "HP implies $0^\sharp$ exists":

The counterexample

- The first step "$Det(\Sigma^1_1)$ implies HP" is provable in Z_2.

Question

Is "HP implies $0^\sharp$ exists" provable in Z_2?

The counterexample I find is the sentence: "HP implies $0^\sharp$ exists":

Theorem

(1) *"HP implies $0^\sharp$ exists" is not provable in Z_2.*

(2) *"HP implies $0^\sharp$ exists" is not provable in Z_3.*

(3) *"HP implies $0^\sharp$ exists" is provable in Z_4.*

So Z_4 is the minimal system in higher order arithmetic to show that "HP implies $0^\sharp$ exists".

Summary of results

Summary of results

- We find an interesting classic mathematical theorem from set theory which is expressible in Z_2 but not provable in Z_2: "HP implies $0^\sharp$ exists".

Summary of results

- We find an interesting classic mathematical theorem from set theory which is expressible in Z_2 but not provable in Z_2: "HP implies $0^\sharp$ exists".
- "HP implies $0^\sharp$ exists" is also not provable in Z_3.

Summary of results

- We find an interesting classic mathematical theorem from set theory which is expressible in Z_2 but not provable in Z_2: "HP implies $0^\sharp$ exists".
- "HP implies $0^\sharp$ exists" is also not provable in Z_3.
- But, in Z_4, HP is equivalent to $0^\sharp$ exists.

Summary of results

- We find an interesting classic mathematical theorem from set theory which is expressible in Z_2 but not provable in Z_2: "HP implies $0^\sharp$ exists".
- "HP implies $0^\sharp$ exists" is also not provable in Z_3.
- But, in Z_4, HP is equivalent to $0^\sharp$ exists.
- Hence, Z_4 is the minimal system in higher order arithmetic to show that HP implies $0^\sharp$ exists.

Summary of results

- We find an interesting classic mathematical theorem from set theory which is expressible in Z_2 but not provable in Z_2: "HP implies $0^\sharp$ exists".
- "HP implies $0^\sharp$ exists" is also not provable in Z_3.
- But, in Z_4, HP is equivalent to $0^\sharp$ exists.
- Hence, Z_4 is the minimal system in higher order arithmetic to show that HP implies $0^\sharp$ exists.

Theorem (joint work with Ralf Schindler)

1. $Z_2 +$ HP *is equiconsistent with* ZFC.
2. $Z_3 +$ HP *is equiconsistent with* ZFC $+$ *there exists a remarkable cardinal.*

Finding the limit of Incompleteness for subsystems of **PA**

Question: Exactly how much information of **PA** is needed for the proof of **G1** and **G2**?

Goal: Finding the limit of Incompleteness for subsystems of **PA**.

- An interpretation of a theory T in a theory S is a mapping from formulas of T to formulas of S that maps all axioms of T to sentences provable in S.
- Let $Int(S)$ denote the degree of interpretation of theory S. $Int(T) < Int(S)$ means that T is interpretable in S but S is not interpretable in T. $Int(T) = Int(S)$ means that T and S are mutually interpretable.
- Interpretability can be accepted as a measure of strength of first order theory.

Definition

Let T be a recursively axiomatizable consistent theory.

1. **G1** *holds for T iff for any recursively axiomatizable consistent theory S, if T is interpretable in S, then S is undecidable.*
2. *T is essentially undecidable iff any recursively axiomatizable consistent extension of T is undecidable.*
3. *T is essentially incomplete iff any recursively axiomatizable consistent extension of T is imcomplete.*

Proposition

Let T be a recursively axiomatizable consistent theory. The followings are equivalent:

1. **G1** *holds for T.*
2. *T is essentially undecidable.*
3. *T is essentially incomplete.*

Robinson's **Q**

Question

Could we find a theory S with minimal degree of interpretation such that **G1** *holds for S?*

Definition

Let Robinson's **Q** *be the system consisting of the following sentences:*

1. $\forall x \forall y(\mathbf{S}x = \mathbf{S}y \rightarrow x = y)$;
2. $\forall x(\mathbf{S}x \neq \mathbf{0})$;
3. $\forall x(x \neq \mathbf{0} \rightarrow \exists y x = \mathbf{S}y)$;
4. $\forall x \forall y(x + \mathbf{0} = x)$;
5. $\forall x \forall y(x + \mathbf{S}y = \mathbf{S}(x + y))$;
6. $\forall x(x \cdot \mathbf{0} = \mathbf{0})$;
7. $\forall x \forall y(x \cdot \mathbf{S}y = x \cdot y + x)$.

System **R**

We work on $L(\overline{0}, \cdots, \overline{n} \cdots, +, \cdot, \leq)$ with infinitely many constants as names for natural numbers and with $\leq$ as primitive symbol.

Definition

Let **R** *be the system consisting of schemes* $Ax1$-$Ax5$ *where* $m, n \in \mathbb{N}$.

$Ax1$ $\overline{m} + \overline{n} = \overline{m+n}$;
$Ax2$ $\overline{m} \neq \overline{n}$ if $m \neq n$;
$Ax3$ $\overline{m} \cdot \overline{n} = \overline{m \cdot n}$;
$Ax4$ $\forall x(x \leq \overline{n} \rightarrow x = \overline{0} \vee \cdots \vee x = \overline{n})$;
$Ax5$ $\forall x(x \leq \overline{n} \vee \overline{n} \leq x)$.

Theorem

(Albert Visser) Suppose T is an R.E. theory. Then T is locally finite (any finite sub-theory of T has a finite model) iff T is interpretable in **R**.

Properties of **Q** and **R**

1. **R** is a sub-theory of **Q**; **Q** is finitely axiomatizable but **R** is not.
2. **Q** is minimal essentially undecidable; **R** is not minimal essentially undecidable.
3. $Int(\mathbf{R}) < Int(\mathbf{Q})$ since **Q** is not interpretable in **R**.

Properties of $\mathbf{Q}$ and $\mathbf{R}$

1. $\mathbf{R}$ is a sub-theory of $\mathbf{Q}$; $\mathbf{Q}$ is finitely axiomatizable but $\mathbf{R}$ is not.
2. $\mathbf{Q}$ is minimal essentially undecidable; $\mathbf{R}$ is not minimal essentially undecidable.
3. $Int(\mathbf{R}) < Int(\mathbf{Q})$ since $\mathbf{Q}$ is not interpretable in $\mathbf{R}$.

Theorem (Folklore)

G1 *holds for* $\mathbf{R}$.

Properties of **Q** and **R**

1. **R** is a sub-theory of **Q**; **Q** is finitely axiomatizable but **R** is not.
2. **Q** is minimal essentially undecidable; **R** is not minimal essentially undecidable.
3. $Int(\mathbf{R}) < Int(\mathbf{Q})$ since **Q** is not interpretable in **R**.

Theorem (Folklore)

G1 *holds for* **R**.

Question

Could we find a theory S such that **G1** *holds for S and $Int(S) < Int(\mathbf{R})$?*

System $\overline{\mathbf{R}}$

Definition

Let $\overline{\mathbf{R}}$ *be the system consisting of schemes* $\Omega_2, \Omega_3, \Omega_4'$ *where* $m, n \in \mathbb{N}$.

Ax2 $\overline{m} \neq \overline{n}$ *if* $m \neq n$;

Ax3 $\overline{m} \cdot \overline{n} = \overline{m \cdot n}$;

Ax′ $\forall x (x \leq \overline{n} \leftrightarrow x = \overline{0} \vee \cdots \vee x = \overline{n})$.

System $\overline{\mathbf{R}}$

Definition

Let $\overline{\mathbf{R}}$ be the system consisting of schemes $\Omega_2, \Omega_3, \Omega_4'$ where $m, n \in \mathbb{N}$.

Ax2 $\overline{m} \neq \overline{n}$ *if* $m \neq n$;

Ax3 $\overline{m} \cdot \overline{n} = \overline{m \cdot n}$;

Ax′ $\forall x(x \leq \overline{n} \leftrightarrow x = \overline{0} \vee \cdots \vee x = \overline{n})$.

$\overline{\mathbf{R}}$ is minimal essentially undecidable: if deleting any axiom of $\overline{\mathbf{R}}$, then the remaining sub-theory is not essentially undecidable.

System $\overline{\mathbf{R}}$

Definition

Let $\overline{\mathbf{R}}$ be the system consisting of schemes $\Omega_2, \Omega_3, \Omega_4'$ where $m, n \in \mathbb{N}$.

Ax2 $\overline{m} \neq \overline{n}$ *if* $m \neq n$;

Ax3 $\overline{m} \cdot \overline{n} = \overline{m \cdot n}$;

Ax′ $\forall x(x \leq \overline{n} \leftrightarrow x = \overline{0} \vee \cdots \vee x = \overline{n})$.

$\overline{\mathbf{R}}$ is minimal essentially undecidable: if deleting any axiom of $\overline{\mathbf{R}}$, then the remaining sub-theory is not essentially undecidable.

Theorem

(1) **G1** *holds for* $\overline{\mathbf{R}}$.

(2) $\mathbf{R}$ *is interpretable in* $\overline{\mathbf{R}}$, *and hence* $Int(\overline{\mathbf{R}}) = Int(\mathbf{R})$.

Definition

$\langle S, T\rangle$ is a recursively inseparable pair if $S, T \subseteq \mathbb{N}$ both are recursively enumerable and there is no recursive set $X \subseteq \mathbb{N}$ such that $S \subseteq X$ and $X \cap T = \emptyset$.

Definition

$\langle S, T\rangle$ is a recursively inseparable pair if $S, T \subseteq \mathbb{N}$ both are recursively enumerable and there is no recursive set $X \subseteq \mathbb{N}$ such that $S \subseteq X$ and $X \cap T = \emptyset$.

Theorem

For any recursively inseparable pair $\langle S, T\rangle$, there exists theory $U_{\langle S,T\rangle}$ such that **G1** *holds for $U_{\langle S,T\rangle}$ and $Int(U_{\langle S,T\rangle}) < Int(\mathbf{R})$.*

Definition

$\langle S, T\rangle$ is a recursively inseparable pair if $S, T \subseteq \mathbb{N}$ both are recursively enumerable and there is no recursive set $X \subseteq \mathbb{N}$ such that $S \subseteq X$ and $X \cap T = \emptyset$.

Theorem

*For any recursively inseparable pair $\langle S, T\rangle$, there exists theory $U_{\langle S,T\rangle}$ such that **G1** holds for $U_{\langle S,T\rangle}$ and $Int(U_{\langle S,T\rangle}) < Int(\mathbf{R})$.*

Definition

Let $\langle S, T\rangle$ be a recursively inseparable pair. Let L be the finite language $\{\mathbf{0}, \mathbf{S}, \mathbf{P}\}$. Consider the following theory $U_{\langle S,T\rangle}$:

- $\overline{m} \neq \overline{n}$ if $m \neq n$;
- $\mathbf{P}(\overline{n})$ if $n \in S$;
- $\neg\mathbf{P}(\overline{n})$ if $n \in T$.

In the following, let $\langle S, T \rangle$ be an arbitrary recursively inseparable pair.

In the following, let $\langle S, T\rangle$ be an arbitrary recursively inseparable pair.

Lemma

G1 *holds for* $U_{\langle S,T\rangle}$.

In the following, let $\langle S, T\rangle$ be an arbitrary recursively inseparable pair.

Lemma

G1 *holds for* $U_{\langle S,T\rangle}$.

Lemma

$U_{\langle S,T\rangle}$ *is interpretable in* **R**.

In the following, let $\langle S, T\rangle$ be an arbitrary recursively inseparable pair.

Lemma

G1 *holds for* $U_{\langle S,T\rangle}$.

Lemma

$U_{\langle S,T\rangle}$ *is interpretable in* **R**.

Theorem

R *is not interpretable in* $U_{\langle S,T\rangle}$.

In the following, let $\langle S, T\rangle$ be an arbitrary recursively inseparable pair.

Lemma

G1 *holds for* $U_{\langle S,T\rangle}$.

Lemma

$U_{\langle S,T\rangle}$ *is interpretable in* $\mathbf{R}$.

Theorem

$\mathbf{R}$ *is not interpretable in* $U_{\langle S,T\rangle}$.

Corollary

G1 *holds for* $U_{\langle S,T\rangle}$ *and* $Int(U_{\langle S,T\rangle}) < Int(\mathbf{R})$.

Model completion of the empty theory

Definition

Model completion of the empty theory

Definition

1. *A consistent theory T is said to be model complete if for all models $\mathfrak{A}$, $\mathfrak{B}$ of T, if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \prec \mathfrak{B}$.*

Model completion of the empty theory

Definition

1. *A consistent theory T is said to be model complete if for all models $\mathfrak{A}$, $\mathfrak{B}$ of T, if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \prec \mathfrak{B}$.*
2. *A theory T^* is a model companion of T if T^* is a cotheory of T and T^* is model complete.*

Model completion of the empty theory

Definition

1. *A consistent theory T is said to be model complete if for all models $\mathfrak{A}$, $\mathfrak{B}$ of T, if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \prec \mathfrak{B}$.*
2. *A theory T^* is a model companion of T if T^* is a cotheory of T and T^* is model complete.*
3. *A theory T^* is a model completion of T if T^* is a model companion of T and for every model $\mathfrak{A}$ of T with diagram $\Delta_{\mathfrak{A}}$, $T^* \cup \Delta_{\mathfrak{A}}$ is complete.*

Model completion of the empty theory

Definition

1. *A consistent theory T is said to be model complete if for all models $\mathfrak{A}$, $\mathfrak{B}$ of T, if $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \prec \mathfrak{B}$.*
2. *A theory T^* is a model companion of T if T^* is a cotheory of T and T^* is model complete.*
3. *A theory T^* is a model completion of T if T^* is a model companion of T and for every model $\mathfrak{A}$ of T with diagram $\Delta_{\mathfrak{A}}$, $T^* \cup \Delta_{\mathfrak{A}}$ is complete.*
4. *Let $\mathcal{K}$ be a class of structures in the same language. A model $M \in \mathcal{K}$ is essentially closed in $\mathcal{K}$ if for any model $N \supseteq M$ such that $N \in \mathcal{K}$, we have every existential formula with parameters from M which is satisfied in N is already satisfied in M.*

For any language L, let EC_L be the model completion of the empty L-theory. Then

For any language L, let EC_L be the model completion of the empty L-theory. Then

Fact

(1) *EC_L has elimination of quantifiers.*

(2) *Models of EC_L are exactly the existentially closed L-structures; in particular, every L-structure embeds in a model of EC_L.*

For any language L, let EC_L be the model completion of the empty L-theory. Then

Fact

(1) *EC_L has elimination of quantifiers.*

(2) *Models of EC_L are exactly the existentially closed L-structures; in particular, every L-structure embeds in a model of EC_L.*

Definition

Consider the following theory S in the language $\langle \in \rangle$ axiomatized by the sentences $\exists z, x_0, \ldots, x_n (\bigwedge_{i<j<n} x_i \neq x_j \wedge \forall y (y \in z \leftrightarrow \bigvee_{i<n} y = x_i))$ for all $n \in \omega$.

Proof of the main theorem

Theorem (Emil JeŘábek)

For any language L and formula $\phi(\overline{z}, \overline{x}, \overline{y})$ with $lh(\overline{x}) = lh(\overline{y})$, there is a constant n with the following property. Let $M \models EC_L$ and $\overline{u} \in M$ be such that $M \models \overline{x}_0, \cdots, \overline{x}_{n-1} \bigwedge_{i<j<n} \phi(\overline{u}, \overline{x}_i, \overline{x}_j)$. Then for every $m \in \omega$ and an asymmetric relation R on $\{0, \cdots, m-1\}$, $M \models \overline{x}_0, \cdots, \overline{x}_{m-1} \bigwedge_{\langle s,t\rangle \in R} \phi(\overline{u}, \overline{x}_s, \overline{x}_t)$.

Proof.

Emil's proof uses Ramsey's theory and indiscernibility argument. □

Proof of the main theorem

Theorem (Emil JeŘábek)

For any language L and formula $\phi(\overline{z}, \overline{x}, \overline{y})$ with $lh(\overline{x}) = lh(\overline{y})$, there is a constant n with the following property. Let $M \models EC_L$ and $\overline{u} \in M$ be such that $M \models \overline{x}_0, \cdots, \overline{x}_{n-1} \bigwedge_{i<j<n} \phi(\overline{u}, \overline{x}_i, \overline{x}_j)$. Then for every $m \in \omega$ and an asymmetric relation R on $\{0, \cdots, m-1\}$, $M \models \overline{x}_0, \cdots, \overline{x}_{m-1} \bigwedge_{\langle s,t\rangle \in R} \phi(\overline{u}, \overline{x}_s, \overline{x}_t)$.

Proof.

Emil's proof uses Ramsey's theory and indiscernibility argument. □

Corollary

S is not weakly interpretable in EC_L (S is not interpretable in any consistent extension of EC_L) for any language L.

Proof of the main theorem: continued

In the following, based on Emil's work I show that **R** is not interpretable in $U_{\langle S,T\rangle}$.

Proof of the main theorem: continued

In the following, based on Emil's work I show that **R** is not interpretable in $U_{\langle S,T\rangle}$.

- Note that S is interpretable in **R**.
- Since S is not weakly interpretable in EC_L for any language L, **R** is not weakly interpretable in EC_L for any language L.

Proof of the main theorem: continued

In the following, based on Emil's work I show that **R** is not interpretable in $U_{\langle S,T\rangle}$.

- Note that S is interpretable in **R**.
- Since S is not weakly interpretable in EC_L for any language L, **R** is not weakly interpretable in EC_L for any language L.

Lemma

If **R** *is interpretable in* $U_{\langle S,T\rangle}$, *then* **R** *is weakly interpretable in* EC_L *for some language* L.

Proof of the main theorem: continued

In the following, based on Emil's work I show that **R** is not interpretable in $U_{\langle S,T\rangle}$.

- Note that S is interpretable in **R**.
- Since S is not weakly interpretable in EC_L for any language L, **R** is not weakly interpretable in EC_L for any language L.

Lemma

If **R** *is interpretable in* $U_{\langle S,T\rangle}$*, then* **R** *is weakly interpretable in* EC_L *for some language* L*.*

Corollary

R *is not interpretable in* $U_{\langle S,T\rangle}$*.*

Questions for future research

Question

Define $\mathbf{D} = \{Int(S) : Int(S) < Int(\mathbf{R})$ *and* **G1** *holds for* $S\}$.

Questions for future research

Question

Define $\mathbf{D} = \{Int(S) : Int(S) < Int(\mathbf{R})$ *and* $\mathbf{G1}$ *holds for* $S\}$.

1. *Is* $(\mathbf{D}, <)$ *well founded?*

Questions for future research

Question

Define $\mathbf{D} = \{Int(S) : Int(S) < Int(\mathbf{R})$ *and* $\mathbf{G1}$ *holds for* $S\}$.

1. *Is* $(\mathbf{D}, <)$ *well founded?*
2. *Are any two elements of* $\mathbf{D}$ *comparable?*

Questions for future research

Question

Define $\mathbf{D} = \{Int(S) : Int(S) < Int(\mathbf{R})$ *and* $\mathbf{G1}$ *holds for* $S\}$.

1. *Is* $(\mathbf{D}, <)$ *well founded?*
2. *Are any two elements of* $\mathbf{D}$ *comparable?*

Conjecture

$(\mathbf{D}, <)$ *is not well founded and has incomparable elements.*

Question

1. *For recursively inseparable pair* $\langle S, T\rangle$ *and* $\langle U, V\rangle$, *what can we say about* $int(U_{\langle S,T\rangle})$ *and* $int(U_{\langle U,V\rangle})$*?*
2. *Could we find a class of recursively inseparable pair* $\langle S_\alpha, T_\alpha\rangle$ *such that the interpretation degree of* $U_{\langle S_\alpha, T_\alpha\rangle}$ *forms a descending chain?*

Reference list

- Yong Cheng and Ralf Schindler, Harrington's Principle in higher order arithmetic, *The Journal of Symbolic Logic*, Volume 80,Issue 02, June 2015, pp 477-489.
- Yong Cheng, Incompleteness for higher order arithmetic: A specific example from Set Theory, Manuscript, to appear.
- Emil JeŘábek, Recursive functions and existentially closed structures, preprint, 2017.
- Visser, Albert. Why the theory R is special. In Neil Tennant (Eds.), Foundational Adventures - Essays in honour of Harvey M. Friedman (pp. 7-24) (17 p.). College Publications, 2014.

Thanks for your attention!