Modal Quantifiers, Potential Infinity, and Yablo sequences

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Outline

Yablo’s paradox

Arithmetization of Yablo sentences

Potentially infinite domains and sl-semantics

Modal interpretation of quantifiers in potentially infinite domains

LYD makes YS msl-fail

Summing up
Yablo’s paradox

\[ Y_0 \quad \text{For any } k > 0, Y_k \text{ is false.} \]
\[ Y_1 \quad \text{For any } k > 1, Y_k \text{ is false.} \]
\[ Y_2 \quad \text{For any } k > 2, Y_k \text{ is false.} \]
\[ \vdots \]
\[ Y_n \quad \text{For any } k > n, Y_k \text{ is false.} \]
\[ \vdots \]
Yablo’s paradox

\[ Y_0 \text{ For any } k > 0, Y_k \text{ is false.} \]
\[ Y_1 \text{ For any } k > 1, Y_k \text{ is false.} \]
\[ Y_2 \text{ For any } k > 2, Y_k \text{ is false.} \]
\[ \vdots \]
\[ Y_n \text{ For any } k > n, Y_k \text{ is false.} \]
\[ \vdots \]

- Suppose \( Y_n \).
Yablo’s paradox

\[ Y_0 \text{ For any } k > 0, \ Y_k \text{ is false.} \]
\[ Y_1 \text{ For any } k > 1, \ Y_k \text{ is false.} \]
\[ Y_2 \text{ For any } k > 2, \ Y_k \text{ is false.} \]
\[ \vdots \]
\[ Y_n \text{ For any } k > n, \ Y_k \text{ is false.} \]
\[ \vdots \]

- Suppose \( Y_n \).
- So for any \( j > n, \ \neg Y_j \).
Suppose $Y_n$.

- So for any $j > n$, $\neg Y_j$.
- So $\neg Y_{n+1}$ and for any $j > n + 1$, $\neg Y_j$. 

Yablo’s paradox

\[ Y_0 \quad \text{For any } k > 0, Y_k \text{ is false.} \]
\[ Y_1 \quad \text{For any } k > 1, Y_k \text{ is false.} \]
\[ Y_2 \quad \text{For any } k > 2, Y_k \text{ is false.} \]
\[ \vdots \]
\[ Y_n \quad \text{For any } k > n, Y_k \text{ is false.} \]
\[ \vdots \]
Yablo’s paradox

\[ Y_0 \]  For any \( k > 0 \), \( Y_k \) is false.

\[ Y_1 \]  For any \( k > 1 \), \( Y_k \) is false.

\[ Y_2 \]  For any \( k > 2 \), \( Y_k \) is false.

\[ \vdots \]

\[ Y_n \]  For any \( k > n \), \( Y_k \) is false.

\[ \vdots \]

- Suppose \( Y_n \).
- So for any \( j > n \), \( \neg Y_j \).
- So \( \neg Y_{n+1} \) and for any \( j > n + 1 \), \( \neg Y_j \).
- So \( Y_{n+1} \). Contradiction.
Yablo’s paradox

$Y_0$ For any $k > 0$, $Y_k$ is false.
$Y_1$ For any $k > 1$, $Y_k$ is false.
$Y_2$ For any $k > 2$, $Y_k$ is false.
  
  ...  
$Y_n$ For any $k > n$, $Y_k$ is false.
  
  ...  

- Suppose $Y_n$.
- So for any $j > n$, $\neg Y_j$.
- So $\neg Y_{n+1}$ and for any $j > n + 1$, $\neg Y_j$.
- So $Y_{n+1}$. Contradiction.
- So $\neg Y_n$ unconditionally.
Yablo’s paradox

$Y_0$ For any $k > 0$, $Y_k$ is false.
$Y_1$ For any $k > 1$, $Y_k$ is false.
$Y_2$ For any $k > 2$, $Y_k$ is false.
   
   $...$

$Y_n$ For any $k > n$, $Y_k$ is false.
   
   $...$

- Suppose $Y_n$.
- So for any $j > n$, $\neg Y_j$.
- So $\neg Y_{n+1}$ and for any $j > n + 1$, $\neg Y_j$.
- So $Y_{n+1}$. Contradiction.
- So $\neg Y_n$ unconditionally.
- So $\exists k > n \ Y_k$. Rinse and repeat.
Background assumptions \hfill (Ketland, 2005)

Uniform disquotation

$$\forall x \ (Y(x) \equiv Tr(Y(x)))$$
Background assumptions (Ketland, 2005)

Uniform disquotation
\[ \forall x \ (Y(x) \equiv Tr(Y(x))) \]

Local disquotation
For any particular \( n \), assume \( Y(\bar{n}) \equiv Tr(Y(\bar{n})) \).
Background assumptions (Ketland, 2005)

Uniform disquotation
\[ \forall x \ (Y(x) \equiv Tr(Y(x))) \]

Local disquotation
For any particular \( n \), assume \( Y(\bar{n}) \equiv Tr(Y(\bar{n})) \).

\( \omega \)-rule
If for any \( n \varphi(\bar{n}) \), derive \( \forall x \varphi(x) \).
Finitistic way out?

The idea

If the world is finite, there are only finitely many Yablo sentences, and the last one is vacuously true.
Finitistic way out?

The idea
If the world is finite, there are only finitely many Yablo sentences, and the last one is vacuously true.

The challenge
Make sense of arithmetic in a formal finitistic setting.
Finitistic way out?

The idea
If the world is finite, there are only finitely many Yablo sentences, and the last one is vacuously true.

The challenge
Make sense of arithmetic in a formal finitistic setting.

The strategy
There could be more things: potential infinity.
Arithmetization of Yablo sentences

Theorem
Let $T$ be a first-order theory in the language $\mathcal{L}_\text{Tr}$, containing $Q$ (nice). Then, for any $\mathcal{L}_\text{Tr}$-formula $\varphi(x,y)$ there is a $\mathcal{L}_\text{Tr}$-formula $\psi(x)$ such that:

$$T \models \psi(x) \equiv \varphi(x, \neg\psi(x)^\dagger).$$

Notation ("quantification over numerals")
$Qx P(\neg\varphi(\dot{x})^\dagger)$, where $Q \in \{\forall, \exists\}$, reads:
For all natural numbers $x$ (there exists a natural number $x$ such that), the result of substituting a numeral denoting $x$ for a variable free in $\varphi$ has property $P$.

Definition (Yablo formula/sentence)
$Y(x)$ is a Yablo formula in $T$ iff

$$T \models \forall x (Y(x) \equiv \forall w > x \neg Tr(\neg Y(\dot{w})^\dagger)).$$

Yablo sentences are of the form $Y(\bar{n})$. 
Existence of Yablo Formulas (Priest, 1997)

Theorem

*If T is nice, there exists a Yablo formula in T.*
Existence of Yablo Formulas (Priest, 1997)

**Theorem**

*If T is nice, there exists a Yablo formula in T.*

**Proof.**

- Let $\varphi(x, y) = \forall w > x \neg Tr(sub(y, \ulcorner y \urcorner, name(w)))$.
- By the Diagonal Lemma, there is a formula $Y(x)$ s.t.:

  $T \vdash Y(x) \equiv \forall w > x \neg Tr(sub(\ulcorner Y(x) \urcorner, \ulcorner y \urcorner, name(w)))$.

- $T \vdash Y(x) \equiv \forall w > x \neg Tr(\ulcorner Y(\hat{w}) \urcorner)$.
ω-inconsistency of Yablo formulas (Ketland, 2005)

Statement

Definition (ω-consistency)

T is ω-consistent iff there is no φ(x) s.t. simultaneously:

\[ \forall n \in \omega \ T \vdash \neg \varphi(n) \]

\[ T \vdash \exists x \varphi(x) \]

T is ω-inconsistent o/w.
ω-inconsistency of Yablo formulas \cite{Ketland2005}

Statement

Definition \((\omega\text{-consistency})\)

T is \(\omega\)-consistent iff there is no \(\varphi(x)\) s.t. simultaneously:
\[
\forall n \in \omega \ T \vdash \neg \varphi(n)
\]
\[
T \vdash \exists x \varphi(x)
\]

T is \(\omega\)-inconsistent o/w.

Definition \((\mathcal{P}A_F)\)

Let \(\mathcal{L}_F\) be standard language extended with \(F\).
\[
\mathcal{P}A_F := \mathcal{P}A \cup \{F(n) \equiv \forall x > n \neg F(x) : n \in \omega\}
\]
**ω-inconsistency of Yablo formulas** (Ketland, 2005)

**Statement**

**Definition (ω-consistency)**

T is ω-consistent iff there is no φ(x) s.t. simultaneously:

\[ \forall n \in \omega \ T \vdash \neg \varphi(n) \]

\[ T \vdash \exists x \varphi(x) \]

T is ω-inconsistent o/w.

**Definition (PA_F)**

Let \( \mathcal{L}_F \) be standard language extended with \( F \).

\[ \text{PA}_F := \text{PA} \cup \{ F(n) \equiv \forall x > n \neg F(x) : n \in \omega \} \]

**Theorem**

\( \text{PA}_F \) is ω-inconsistent.
\(\omega\)-inconsistency of Yablo formulas (Ketland, 2005)

Proof

- Work in \(\text{PA}_F\). Fix an \(n \in \omega\) and assume \(F(n)\).
  \[\forall x > n \neg F(x).\]  \(\text{(★)}\)
$\omega$-inconsistency of Yablo formulas (Ketland, 2005)

Proof

- Work in $\text{PA}_F$. Fix an $n \in \omega$ and assume $F(\bar{n})$.
  \[
  \forall x > \bar{n} \neg F(x).
  \]  
  (★)

- In particular, $\forall x > n + 1 \neg F(x)$. 

This is equivalent to $F(n + 1)$. But from (★), $\neg F(n + 1)$ follows. Contradiction.
So unconditionally $\neg F(n)$:
\[
\forall n \in \omega \quad \text{PA}_F \vdash \neg F(n).
\] (1)

By definition of $\text{PA}_F$:
\[
\forall n \in \omega \quad \text{PA}_F \vdash \exists x > n F(x).
\] (2)

(1) + (2) $\Rightarrow$ $\omega$-inconsistency.
\(\omega\)-inconsistency of Yablo formulas (Ketland, 2005)

Proof

- Work in PA\(_F\). Fix an \(n \in \omega\) and assume \(F(\overline{n})\).
  \[
  \forall x > \overline{n} \neg F(x). \tag{★}
  \]

- In particular, \(\forall x > \overline{n + 1} \neg F(x)\).
- This is equivalent to \(F(\overline{n + 1})\).
\[ \omega \text{-inconsistency of Yablo formulas (Ketland, 2005)} \]

**Proof**

- Work in \( \text{PA}_F \). Fix an \( n \in \omega \) and assume \( F(\overline{n}) \).
  \[
  \forall x > \overline{n} \neg F(x). \tag{\star}
  \]

- In particular, \( \forall x > \overline{n + 1} \neg F(x) \).
- This is equivalent to \( F(\overline{n + 1}) \).
- But from (\star), \( \neg F(\overline{n + 1}) \) follows. Contradiction.
ω-inconsistency of Yablo formulas  (Ketland, 2005)

Proof

- Work in $\text{PA}_F$. Fix an $n \in \omega$ and assume $F(\bar{n})$.
  \[
  \forall x > \bar{n} \neg F(x). \tag{★}
  \]

- In particular, $\forall x > \bar{n} + 1 \neg F(x)$.
- This is equivalent to $F(\bar{n} + 1)$.
- But from (★), $\neg F(\bar{n} + 1)$ follows. Contradiction.
- So unconditionally $\neg F(\bar{n})$: 

\( \omega \)-inconsistency of Yablo formulas \ (Ketland, 2005)

Proof

- Work in \( \text{PA}_F \). Fix an \( n \in \omega \) and assume \( F(\bar{n}) \).
  \[
  \forall x > \bar{n} \neg F(x).
  \tag{\star}
  \]

- In particular, \( \forall x > n + 1 \neg F(x) \).
- This is equivalent to \( F(n + 1) \).
- But from (\( \star \)), \( \neg F(n + 1) \) follows. Contradiction.
- So unconditionally \( \neg F(\bar{n}) \):
  \[
  \forall n \in \omega \text{ } \text{PA}_F \vdash \neg F(\bar{n}).
  \tag{1}
  \]
\( \omega \)-inconsistency of Yablo formulas \textit{(Ketland, 2005)}

Proof

- Work in \( \text{PA}_F \). Fix an \( n \in \omega \) and assume \( F(\overline{n}) \).
  \[
  \forall x > \overline{n} \neg F(x).
  \]
  \( \text{(★)} \)

- In particular, \( \forall x > \overline{n} + 1 \neg F(x) \).
- This is equivalent to \( F(\overline{n} + 1) \).
- But from (★), \( \neg F(\overline{n} + 1) \) follows. Contradiction.
- So unconditionally \( \neg F(\overline{n}) \):
  \[
  \forall n \in \omega \text{ } \text{PA}_F \vdash \neg F(\overline{n}).
  \]
  \( \text{(1)} \)

- By definition of \( \text{PA}_F \):
  \[
  \forall n \in \omega \text{ } \text{PA}_F \vdash \exists x > \overline{n} F(x).
  \]
ω-inconsistency of Yablo formulas (Ketland, 2005)

Proof

- Work in $\text{PA}_F$. Fix an $n \in \omega$ and assume $F(\bar{n})$.
  $$\forall x > \bar{n} \neg F(x). \quad (\star)$$

- In particular, $\forall x > n + 1 \neg F(x)$.
- This is equivalent to $F(n + 1)$.
- But from $(\star)$, $\neg F(n + 1)$ follows. Contradiction.
- So unconditionally $\neg F(\bar{n})$:
  $$\forall n \in \omega \text{ PA}_F \vdash \neg F(\bar{n}). \quad (1)$$

- By definition of $\text{PA}_F$:
  $$\forall n \in \omega \text{ PA}_F \vdash \exists x > \bar{n} F(x). \quad (2)$$
**ω*-inconsistency of Yablo formulas* (Ketland, 2005)

**Proof**

- Work in $\text{PA}_F$. Fix an $n \in \omega$ and assume $F(\bar{n})$.
  
  \[ \forall x > \bar{n} \neg F(x). \]  
  \[ (\star) \]

- In particular, $\forall x > \bar{n} + 1 \neg F(x)$.

- This is equivalent to $F(\bar{n} + 1)$.

- But from $(\star)$, $\neg F(\bar{n} + 1)$ follows. Contradiction.

- So unconditionally $\neg F(\bar{n})$:
  
  \[ \forall n \in \omega \quad \text{PA}_F \vdash \neg F(\bar{n}). \]  
  \[ (1) \]

- By definition of $\text{PA}_F$:
  
  \[ \forall n \in \omega \quad \text{PA}_F \vdash \exists x > \bar{n} F(x). \]

- In particular:
  
  \[ \text{PA}_F \vdash \exists x F(x). \]  
  \[ (2) \]

- $(1) + (2) \Rightarrow \omega$-inconsistency.
The consistency of Yablo formulas

Theorem

$PA_F$ is consistent.
The consistency of Yablo formulas

**Theorem**

$\text{PA}_F$ is consistent.

**Proof.**

- Take a nonstandard model $\mathcal{M}$ of PA.

But also, $(\mathcal{M}, A) \models \exists x F(x)$.

Moreover, $\forall n \in \omega (\mathcal{M}, A) \models \exists x > n F(x)$.

Hence $\forall n \in \omega (\mathcal{M}, A) \models F(n) \equiv \forall x > n \neg F(x)$ (both sides are false).

So $(\mathcal{M}, A) \models \text{PA}_F$ and $	ext{PA}_F$ is consistent.
The consistency of Yablo formulas

Theorem

$\text{PA}_F$ is consistent.

Proof.

- Take a nonstandard model $M$ of PA.
- Pick a nonstandard $a \in M$, let $A = \{a\}$. 

But also, $(M, A) \models \exists x F(x)$.

Moreover, $\forall n \in \omega (M, A) \models \exists x > n F(x)$.

Hence $\forall n \in \omega (M, A) \models F(n) \equiv \forall x > n \neg F(x)$ (both sides are false).

So $(M, A) \models \text{PA}_F$ and $\text{PA}_F$ is consistent.
The consistency of Yablo formulas

Theorem

PA_F is consistent.

Proof.

• Take a nonstandard model \( \mathcal{M} \) of PA.
• Pick a nonstandard \( a \in M \), let \( A = \{a\} \).
• Put \( F^M = A \).
The consistency of Yablo formulas

Theorem

$\text{PA}_F$ is consistent.

Proof.

- Take a nonstandard model $\mathcal{M}$ of PA.
- Pick a nonstandard $a \in M$, let $A = \{a\}$.
- Put $F^\mathcal{M} = A$.
- $\forall n \in \omega \ (\mathcal{M}, A) \models \neg F(n)$. 
The consistency of Yablo formulas

Theorem

$\text{PA}_F$ is consistent.

Proof.

- Take a nonstandard model $\mathcal{M}$ of $\text{PA}$.
- Pick a nonstandard $a \in M$, let $A = \{a\}$.
- Put $F^\mathcal{M} = A$.
- $\forall n \in \omega \ (\mathcal{M}, A) \models \neg F(n)$.
- But also, $(\mathcal{M}, A) \models \exists x F(x)$. 
The consistency of Yablo formulas

Theorem

$\text{PA}_F$ is consistent.

Proof.

- Take a nonstandard model $\mathcal{M}$ of PA.
- Pick a nonstandard $a \in M$, let $\mathcal{A} = \{a\}$.
- Put $F^\mathcal{M} = \mathcal{A}$.
- $\forall n \in \omega \ (\mathcal{M}, \mathcal{A}) \models \neg F(n)$.
- But also, $(\mathcal{M}, \mathcal{A}) \models \exists x \ F(x)$.
- Moreover, $\forall n \in \omega \ (\mathcal{M}, \mathcal{A}) \models \exists x > n \ F(x)$. 
The consistency of Yablo formulas

Theorem

$\text{PA}_F$ is consistent.

Proof.

- Take a nonstandard model $\mathcal{M}$ of PA.
- Pick a nonstandard $a \in M$, let $A = \{a\}$.
- Put $F^M = A$.
- $\forall n \in \omega \ (\mathcal{M}, A) \models \neg F(n)$.
- But also, $(\mathcal{M}, A) \models \exists x \ F(x)$.
- Moreover, $\forall n \in \omega \ (\mathcal{M}, A) \models \exists x > n \ F(x)$.
- Hence $\forall n \in \omega \ (\mathcal{M}, A) \models F(n) \equiv \forall x > n \neg F(x)$ (both sides are false).
The consistency of Yablo formulas

Theorem

$\text{PA}_F$ is consistent.

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- Take a nonstandard model $\mathcal{M}$ of PA.
- Pick a nonstandard $a \in M$, let $A = \{a\}$.
- Put $F^\mathcal{M} = A$.
- $\forall n \in \omega \ (\mathcal{M}, A) \models \neg F(n)$.
- But also, $(\mathcal{M}, A) \models \exists x \ F(x)$.
- Moreover, $\forall n \in \omega \ (\mathcal{M}, A) \models \exists x > n \ F(x)$.
- Hence $\forall n \in \omega \ (\mathcal{M}, A) \models F(n) \equiv \exists x > n \neg F(x)$
  (both sides are false).
- So $(\mathcal{M}, A) \models \text{PA}_F$ and $\text{PA}_F$ is consistent.
Adding local disquotation

Definition

\[ AD = \{ \text{Tr}(\overline{\phi}) \equiv \phi : \phi \in \text{Sent}_L \} \]

\[ YD = \{ \text{Tr}(\overline{Y(n)}) \equiv Y(n) : Y(n) \text{ belongs to the Yablo sequence} \} \]
Adding local disquotation

**Definition**

\[ AD = \{ \text{Tr}(\overline{\varphi}) \equiv \varphi : \varphi \in \text{Sent}_L \} \]

\[ YD = \{ \text{Tr}(\overline{Y(n)}) \equiv Y(n) : Y(n) \text{ belongs to the Yablo sequence} \}. \]

**Definition (PA_D)**

PAT is obtained from PA by adding Tr (induction!)
\[ PA_D = PAT \cup AD \cup YD. \]
\[ PA_D^- \text{ is } PA_D \text{ with induction without } Tr. \]
Adding local disquotation

Definition

\[ AD = \{ \text{Tr}(\neg \neg \varphi) \equiv \varphi : \varphi \in \text{Sent}_L \} \]

\[ YD = \{ \text{Tr}(\neg Y(n)) \equiv Y(n) : Y(n) \text{ belongs to the Yablo sequence} \} \]

Definition (\(\text{PA}_D\))

\(\text{PAT}\) is obtained from \(\text{PA}\) by adding \(\text{Tr}\) (induction!)

\(\text{PA}_D = \text{PAT} \cup AD \cup YD.\)

\(\text{PA}_D^-\) is \(\text{PA}_D\) with induction without \(\text{Tr}\).

Theorem

\(\text{PA}_D\) is \(\omega\)-inconsistent.
Adding local disquotation

**Theorem**

$\mathsf{PA}_D$ is $\omega$-inconsistent.

**Proof.**

- Existence of $\mathsf{YF}$ entails:

  \[
  \forall n \in \omega \ \mathsf{PA}_D \vdash Y(n) \equiv \forall x > n \neg \mathsf{Tr}(\llbracket Y(x) \rrbracket) .
  \]

So $\mathsf{PA}_D$ contains $\mathsf{PA}_F$ (which is $\omega$-inconsistent).
Adding local disquotation

Theorem
\( \text{PA}_D \) is \( \omega \)-inconsistent.

Proof.
- Existence of \( YF \) entails:
  \[ \forall n \in \omega \ \text{PA}_D \vdash Y(n) \equiv \forall x > n \neg \text{Tr}(\Gamma Y(x)). \]
- By the inclusion of \( YD \) we get:
  \[ \forall n \in \omega \ \text{PA}_D \vdash \text{Tr}(\Gamma Y(n)) \equiv \forall x > n \neg \text{Tr}(\Gamma Y(x)). \]
Adding local disquotation

Theorem

$\text{PA}_D$ is $\omega$-inconsistent.

Proof.

- Existence of $\text{YF}$ entails:
  \[
  \forall n \in \omega \quad \text{PA}_D \vdash Y(n) \equiv \forall x > n \rightarrow \text{Tr}(\overline{\text{Y}(\dot{x})})\].
- By the inclusion of $\text{YD}$ we get:
  \[
  \forall n \in \omega \quad \text{PA}_D \vdash \text{Tr}(\overline{\text{Y}(n)}) \equiv \forall x > n \rightarrow \text{Tr}(\overline{\text{Y}(\dot{x})})\].
- Let $F(x) := \text{Tr}(\overline{\text{Y}(\dot{x})})$:
  \[
  \forall n \in \omega \quad \text{PA}_D \vdash F(n) \equiv \forall x > n \rightarrow F(x)\].
Adding local disquotation

Theorem

$\text{PA}_D$ is $\omega$-inconsistent.

Proof.

- Existence of $YF$ entails:
  $$\forall n \in \omega \text{ } \text{PA}_D \vdash Y(\bar{n}) \equiv \forall x > \bar{n} \neg \text{Tr}(\bar{\text{Y} (\bar{x})})$$

- By the inclusion of $YD$ we get:
  $$\forall n \in \omega \text{ } \text{PA}_D \vdash \text{Tr}(\bar{\text{Y} (\bar{n})}) \equiv \forall x > \bar{n} \neg \text{Tr}(\bar{\text{Y} (\bar{x})})$$

- Let $F(x) := \text{Tr}(\bar{\text{Y} (\bar{x})})$:
  $$\forall n \in \omega \text{ } \text{PA}_D \vdash F(\bar{n}) \equiv \forall x > \bar{n} \neg F(x)$$

- So $\text{PA}_D$ contains $\text{PA}_F$ (which is $\omega$-inconsistent).
The consistency of $\mathsf{PA}_D^-$

**Theorem**

$\mathsf{PA}_D^-$ is consistent.
The consistency of $\mathsf{PA}_D^-$

**Theorem**

$\mathsf{PA}_D^-$ is consistent.

**Proof.**

- Take a nonstandard $\mathcal{M}$ of $\mathsf{PA}$.
The consistency of $\text{PA}_D^-$

**Theorem**

$\text{PA}_D^-$ is consistent.

**Proof.**

- Take a nonstandard $\mathcal{M}$ of $\text{PA}$.
- Let $t(x) := \upharpoonright Y(x) \upharpoonright$. By overspill, there are nonstandard $b$ and $c$ such that $t^\mathcal{M}(b) = c$. 
The consistency of $\mathsf{PA_D}^-$

**Theorem**

$\mathsf{PA_D}^-$ is consistent.

**Proof.**

- Take a nonstandard $\mathcal{M}$ of $\mathsf{PA}$.
- Let $t(x) := \neg Y(\dot{x})^\upharpoonright$. By overspill, there are nonstandard $b$ and $c$ such that $t^\mathcal{M}(b) = c$.
- Let $\text{Tr}^\mathcal{M} = S = \text{Th}_\mathcal{L}(\mathcal{M}) \cup \{c\}$. Clearly, $(\mathcal{M}, S) \models AD$.
  \[
  \forall n \in \omega \ (\mathcal{M}, S) \models \exists x > n \ \text{Tr}(\neg Y(\dot{x})^\upharpoonright)
  \]
  \[
  \forall n \in \omega \ (\mathcal{M}, S) \models \neg Y(\bar{n})
  \]
The consistency of $\text{PA}_D^-$

Theorem

$\text{PA}_D^-$ is consistent.

Proof.

- Take a nonstandard $\mathcal{M}$ of PA.
- Let $t(x) := \neg Y(\dot{x})$. By overspill, there are nonstandard $b$ and $c$ such that $t^\mathcal{M}(b) = c$.
- Let $Tr^\mathcal{M} = S = \text{Th}_\mathcal{L}(\mathcal{M}) \cup \{c\}$. Clearly, $(\mathcal{M}, S) \models AD$.
  - $\forall n \in \omega \ (\mathcal{M}, S) \models \exists x > n \ Tr(\neg Y(\dot{x}))$
  - $\forall n \in \omega \ (\mathcal{M}, S) \models \neg Y(\bar{n})$
- Standard $Y(n)$ are not in $S$, so:
  - $\forall n \in \omega \ (\mathcal{M}, S) \models \neg Tr(\neg Y(\bar{n}))$. 

So $(\mathcal{M}, S) \models YD$ (UYD fails here).
The consistency of $\text{PA}_D^-$

**Theorem**

$\text{PA}_D^-$ *is consistent.*

**Proof.**

- Take a nonstandard $\mathcal{M}$ of PA.
- Let $t(x) := \forall Y(\dot{x}) \uparrow$. By overspill, there are nonstandard $b$ and $c$ such that $t^\mathcal{M}(b) = c$.
- Let $\text{Tr}^\mathcal{M} = S = \text{Th}_L(\mathcal{M}) \cup \{c\}$. Clearly, $(\mathcal{M}, S) \models \text{AD}$.
  - $\forall n \in \omega$ $(\mathcal{M}, S) \models \exists x > n$ $\text{Tr}(\forall Y(\dot{x}) \uparrow)$
  - $\forall n \in \omega$ $(\mathcal{M}, S) \models \neg Y(\bar{n})$
- Standard $Y(n)$ are not in $S$, so:
  - $\forall n \in \omega$ $(\mathcal{M}, S) \models \neg \text{Tr}(\forall Y(\bar{n}) \uparrow)$.
- So $(\mathcal{M}, S) \models \text{YD}$ (UYD fails here).
The consistency of $\text{PA}_D$

**Theorem**

$\text{PA}_D$ is consistent.
The consistency of $\text{PA}_D$

**Theorem**

$\text{PA}_D$ is consistent.

**Proof.**

By finite satisfiability (put only the last Yablo sentence in the extension of $Tr$, check induction holds), and compactness.
Conservativeness of $\mathsf{PA}_D$

Theorem

$\mathsf{PA}_D$ is a conservative extension of $\mathsf{PA}$. 

Proof. Suppose $\mathsf{PA} \not\vdash \phi$. So $\mathsf{PA} \cup \{\neg \phi\}$ is consistent. For a nonstandard $M$ of $\mathsf{PA}$, $M \models \neg \phi$. There is an elementarily equivalent $M'$ such that $(M', \text{Tr}_{M'}) \models \mathsf{PA}_D$. So $(M', \text{Tr}_{M'}) \not\models \phi$, and so $\mathsf{PA}_D \not\vdash \phi$. 


Conservativeness of PA$_D$

Theorem

PA$_D$ is a conservative extension of PA.

Proof.

1. Suppose PA $\vdash \varphi$.
Conservativeness of $\text{PA}_D$

**Theorem**

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**Proof.**

- Suppose $\text{PA} \vdash \varphi$.
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Conservativeness of \( \text{PA}_D \)

**Theorem**

\( \text{PA}_D \) is a conservative extension of \( \text{PA} \).

**Proof.**

- Suppose \( \text{PA} \not\vdash \varphi \).
- So \( \text{PA} \cup \{\neg \varphi\} \) is consistent.
- For a nonstandard \( \mathcal{M} \) of PA, \( \mathcal{M} \models \neg \varphi \).
Conservativeness of $\text{PA}_D$
Conservativeness of \( \text{PA}_D \)

**Theorem**

\( \text{PA}_D \) is a conservative extension of \( \text{PA} \).

**Proof.**

- Suppose \( \text{PA} \not\vdash \varphi \).
- So \( \text{PA} \cup \{\neg \varphi\} \) is consistent.
- For a nonstandard \( M \) of \( \text{PA} \), \( M \models \neg \varphi \).
- There is an elementarily equivalent \( M' \) such that \( (M', \text{Tr}^{M'}) \models \text{PA}_D \).
- \( (M', \text{Tr}^{M'}) \not\models \varphi \), and so \( \text{PA}_D \not\vdash \varphi \).
Uniform Yablo Disquotation yields contradiction

Definition

\[ UYD = \forall x (Tr(\overline{\neg Y(\dot{x})}) \equiv Y(x)) \]
Uniform Yablo Disquotation yields contradiction

Definition
\[ UYD = \forall x (Tr(\overline{\exists Y(x)}) \equiv Y(x)) \]

Theorem
Let \( S = \text{PAT} + UYD \). \( S \) is inconsistent.
Uniform Yablo Disquotation yields contradiction

Definition

\[ UYD = \forall x (\overline{Tr(\overline{\neg Y(x)})}) \equiv Y(x)) \]

Theorem

Let \( S = \text{PAT} + UYD \). \( S \) is inconsistent.

Work in \( S \).
Uniform Yablo Disquotation yields contradiction

**Definition**

\[ UYD = \forall x (\text{Tr}(\Gamma Y(\dot{x})) \equiv Y(x)) \]

**Theorem**

*Let \( S = \text{PAT} + UYD \). \( S \) is inconsistent.*

**Work in \( S \).**

* \( \forall x (Y(x) \equiv \forall w > x \neg \text{Tr}(\Gamma Y(\dot{w}))) \) [Yablo existence]
Uniform Yablo Disquotation yields contradiction

Definition

\[ UYD = \forall x (\overline{\text{Tr}}(\overline{\overline{Y(x)}}) \equiv Y(x)) \]

Theorem

Let \( S = \text{PAT} + UYD \). \( S \) is inconsistent.

Work in \( S \).

- \( \forall x (Y(x) \equiv \forall w > x \neg \overline{\text{Tr}}(\overline{\overline{Y(w)}})) \) [Yablo existence]
- UYD gives \( \forall x (Y(x) \equiv \forall w > x \neg Y(w)) \).
Uniform Yablo Disquotation yields contradiction

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\[ UYD = \forall x (Tr(\neg \neg \neg \neg \neg Y(\dot{x})) \equiv Y(x)) \]

Theorem

Let \( S = \text{PAT} + \text{UYD} \). \( S \) is inconsistent.

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1. \( \forall x (Y(x) \equiv \forall w > x \neg Tr(\neg \neg \neg \neg \neg Y(\dot{w})) \) [Yablo existence]
2. UYD gives \( \forall x (Y(x) \equiv \forall w > x \neg Y(w)) \).
3. So \( \forall x (Y(x) \equiv \forall w > x \exists z > w Tr(\neg \neg \neg \neg \neg Y(\dot{z})) \) [unraveling].
Uniform Yablo Disquotation yields contradiction

Definition

\[ UYD = \forall x (\overline{\text{Tr}(Y(\dot{x}))} \equiv Y(x)) \]

Theorem

Let \( S = \text{PAT} + UYD \). \( S \) is inconsistent.

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- \( \forall x (Y(x) \equiv \forall w > x \neg \overline{\text{Tr}(Y(\dot{w}))}) \) [Yablo existence]
- UYD gives \( \forall x (Y(x) \equiv \forall w > x \neg Y(w)) \).
- So \( \forall x (Y(x) \equiv \forall w > x \exists z > w \overline{\text{Tr}(Y(\dot{z}))}) \) [unraveling].
- By UYD: \( \forall x (Y(x) \equiv \forall w > x \exists z > w Y(z)) \)
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Definition

\[ UYD = \forall x (Tr(\neg \neg Y(x)) \equiv Y(x)) \]

Theorem

Let \( S = \text{PAT} + \text{UYD} \). \( S \) is inconsistent.

Work in \( S \).

\begin{itemize}
  \item \( \forall x (Y(x) \equiv \forall w > x \neg Tr(\neg \neg Y(\check{w})) \) \ [Yablo existence]
  \item UYD gives \( \forall x (Y(x) \equiv \forall w > x \neg Y(w)) \).
  \item So \( \forall x (Y(x) \equiv \forall w > x \exists z > w \ Tr(\neg \neg Y(\check{z}))) \) \ [unraveling].
  \item By UYD: \( \forall x (Y(x) \equiv \forall w > x \exists z > w \ Y(z)) \)
  \item So \( \forall x (Y(x) \equiv \exists w > x \ Y(w)) \)
\end{itemize}
Uniform Yablo Disquotation yields contradiction

Definition

\[ \text{UYD} = \forall x (\text{Tr}(\neg Y(x)) \equiv Y(x)) \]

Theorem

Let \( S = \text{PAT} + \text{UYD} \). \( S \) is inconsistent.

Work in \( S \).

- \( \forall x (Y(x) \equiv \forall w > x \neg \text{Tr}(\neg Y(w))) \) [Yablo existence]
- UYD gives \( \forall x (Y(x) \equiv \forall w > x \neg Y(w)) \).
- So \( \forall x (Y(x) \equiv \forall w > x \exists z > w \text{Tr}(\neg Y(z))) \) [unraveling].
- By UYD: \( \forall x (Y(x) \equiv \forall w > x \exists z > w Y(z)) \)
- So \( \forall x (Y(x) \equiv \exists w > x Y(w)) \)
- \( \forall x ((\forall w > x \neg Y(w)) \equiv (\exists w > x Y(w))) \)
Local disquotation with $\omega$-rule is inconsistent

**Theorem**

Let $\text{PA}_{D}^{\omega} = (\text{PAT}^- \cup \text{AD} \cup \text{YD})^\omega$. $\text{PA}_{D}^{\omega}$ is inconsistent.

($\text{AD}$ is not needed.)
Local disquotation with \( \omega \)-rule is inconsistent

**Theorem**

Let \( \text{PA}_D^{\omega-} = (\text{PAT}^- \cup \text{AD} \cup \text{YD})^\omega \). \( \text{PA}_D^{\omega-} \) is inconsistent.

(AD is not needed.)

**Proof idea.**

- \( \forall n \in \omega \) \( \text{PA}_D^{\omega-} \vdash \neg Y(\bar{n}) \) [internalized standard reasoning]
Local disquotation with $\omega$-rule is inconsistent

**Theorem**

Let $\text{PA}_D^{\omega^-} = (\text{PAT}^- \cup \text{AD} \cup \text{YD})^\omega$. $\text{PA}_D^{\omega^-}$ is inconsistent.

(AD is not needed.)

**Proof idea.**

- $\forall n \in \omega \; \text{PA}_D^{\omega^-} \vdash \neg Y(n)$ [internalized standard reasoning]
- $\forall n \in \omega \; \text{PA}_D^{\omega^-} \vdash \neg Tr(\overline{\neg Y(n)})$ [Y disquotation]

\[ PA \wedge \overline{\text{Y}(23)} \]
Local disquotation with \( \omega \)-rule is inconsistent

**Theorem**

Let \( \text{PA}_{D}^{\omega-} = (\text{PA}^{\omega}_T \cup \text{AD} \cup \text{YD})^\omega \). \( \text{PA}_{D}^{\omega-} \) is inconsistent.

(AD is not needed.)

**Proof idea.**

- \( \forall n \in \omega \quad \text{PA}_{D}^{\omega-} \vdash \neg Y(n) \) [internalized standard reasoning]
- \( \forall n \in \omega \quad \text{PA}_{D}^{\omega-} \vdash \neg Tr(\Gamma Y(n)) \) [\( Y \) disquotation]
- \( \text{PA}_{D}^{\omega-} \vdash \forall x \neg Tr(\Gamma Y(x)) \) [\( \omega \)-rule]
Local disquotation with $\omega$-rule is inconsistent

**Theorem**

Let $\text{PA}_D^{\omega^-} = (\text{PAT}^- \cup \text{AD} \cup \text{YD})^{\omega}$. $\text{PA}_D^{\omega^-}$ is inconsistent.

(AD is not needed.)

**Proof idea.**

- $\forall n \in \omega \; \text{PA}_D^{\omega^-} \vdash \neg Y(n)$ [internalized standard reasoning]
- $\forall n \in \omega \; \text{PA}_D^{\omega^-} \vdash \neg \text{Tr}(\overline{\forall Y(n)})$ [Y disquotation]
- $\text{PA}_D^{\omega^-} \vdash \forall x \neg \text{Tr}(\overline{\forall Y(x)})$ [$\omega$-rule]
- In particular: $\text{PA}_D^{\omega^-} \vdash Y(\overline{23})$
Classical set-up vs. Yablo

- Even those theories which prove the existence of Yablo sentences are still consistent.
- They’re $\omega$-inconsistent with local Yablo disquotation, though.
- One way to obtain a contradiction: uniform Yablo disquotation.
- Another one: local disquotation and $\omega$ – rule.
sl-semantics (Mostowski, 2001a,b, 2016)

Definition (FM-domains)

Take a relational arithmetical language.

\[ FM(\mathbb{N}) = \{ \mathbb{N}_n : n = 1, 2, ... \} \]

\[ \mathbb{N}_n = (\{0, 1, ..., n - 1\}, +^{(n)}, \times^{(n)}, 0^{(n)}, s^{(n)}, <^{(n)}) \]
**sl-semantics** (Mostowski, 2001a,b, 2016)

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**Definition (sl-theory of FM(\mathbb{N}))**

- Satisfaction in finite points in \( FM(\mathbb{N}) \) is standard.
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- \( sl(FM(\mathbb{N})) = \{ \varphi \in \text{Sent}_L : FM(\mathbb{N}) \models_{sl} \varphi \} \)

Definition (FM(\mathbb{N})^T)

An \( FM(\mathbb{N})^T \)-domain is a set of \( (\mathbb{N}_k, T_k) \) containing a unique member for each \( k \in \omega \), where \( T_k \subset \{0, \ldots, k - 1\} \).
Things that are kinda the same

Syntax is still representable.
Truth is still undefinable.
Diagonal lemma still holds.

Theorem (sl-Yablo existence)
There exists a formula $Y(x)$ s.t. for any FM $(N)$ $\mathcal{T}$-domain:

$\forall n \in \omega \ FM(N) \ 
\mathcal{T}| \equiv \forall x (x > n \Rightarrow \neg \text{Tr}(\langle Y(\dot{x}) \rangle) )$
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**Theorem (sl-Yablo existence)**

There exists a formula $Y(x)$ s.t. for any $FM(\mathbb{N})^T$-domain:

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YS are non-trivially false in the limit

Theorem

For any class $\mathcal{K}$ of finite models, if $\mathcal{K} \models_{sl} AD + YD$, then:

$\forall n \in \omega \mathcal{K} \models_{sl} \neg Y(n)$. AD is not essential.
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Reason.

The standard argument still flies, \textit{mutatis mutandis}.
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Reason.
The standard argument still flies, *mutatis mutandis*.

Theorem
There is an FM-domain $sl$-satisfying AD $\cup$ YD.

Reason.
In each point take truth to refer to all existing codes of true arithmetical formula, and the code of the last YS.
There is no free lunch

Theorem

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Theorem

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- Each particular YS is sl-fails.
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- Some YS is true sl-holds.
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The sl-theory of this model is ω-inconsistent.

Reason.
- Each particular YS is sl-fails.
- In each finite point, the last YS is satisfied.
- Some YS is true sl-holds.

Fact (Cheap shot)
- n is the greatest number sl-fails, for any n.
There is no free lunch

Theorem

There is an $FM$-domain $sl$-satisfying $AD \cup YD$.

Theorem (The cost)

The $sl$-theory of this model is $\omega$-inconsistent.

Reason.
- Each particular YS is $sl$-fails.
- In each finite point, the last YS is satisfied.
- Some YS is true $sl$-holds.

Fact (Cheap shot)
- $n$ is the greatest number $sl$-fails, for any $n$.
- The greatest number exists $sl$-holds.
Modal interpretation of quantifiers (Urbaniak, 2016)

Definition (Accessibility relation in FM-domains)

\[ R(M, N) \text{ iff } M \subseteq N. \]

For \( \mathbb{N}_m, \mathbb{N}_n \in FM(\mathbb{N}) \) this boils down to \( m \leq n \).
Modal interpretation of quantifiers (Urbaniak, 2016)

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Definition (\( m \)-semantics)

- If \( \varphi \) is atomic, then \( (\mathcal{K}, M) \models_m \varphi \), iff \( M \models \varphi \).
- Clauses for boolean connectives are standard.
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Definition (\( msl \)-theory)

\[ msl(\text{FM}(\mathbb{N})) = \{ \varphi : \exists n \forall k k \geq n \Rightarrow (\text{FM}(\mathbb{N}), \mathbb{N}_k) \models_m \varphi \} \]
Modal interpretation of quantifiers (Urbaniaik, 2016)

**Definition (Accessibility relation in FM-domains)**

\[ R(M, N) \text{ iff } M \subseteq N. \]

For \( \mathbb{I} \mathbb{N}_m, \mathbb{I} \mathbb{N}_n \in FM(\mathbb{I} \mathbb{N}) \) this boils down to \( m \leq n \).

**Definition (\( m \)-semantics)**

- If \( \varphi \) is atomic, then \( (\mathcal{K}, M) \models_m \varphi \), iff \( M \models \varphi \).
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**Definition (\( msl \)-theory)**

\[
msl(FM(\mathbb{I} \mathbb{N})) = \{ \varphi : \exists n \ \forall k \ k \geq n \Rightarrow (FM(\mathbb{I} \mathbb{N}), \mathbb{I} \mathbb{N}_k) \models_m \varphi \}
\]

**Example**

\[
(\exists x \ \forall y \ x \geq y) \in sl(FM(\mathbb{I} \mathbb{N})), \notin msl(FM(\mathbb{I} \mathbb{N}))
\]
Arithmetic regained

**Theorem**

\[ \text{msl}(FM(\mathbb{N})) = Th(\mathbb{N}) \]

**Reason.**

- Finite points are submodels of \( \mathbb{N} \).
- Q-free \( \varphi \) are preserved for parameters in a point.
- \( \exists x \varphi \) true in \( \mathbb{N} \) has a finite witness, which belongs to some finite point.
YS & the modal interpretation

Theorem

If $YD \subseteq msl(FM(\mathbb{N})^\gamma)$, then:

$$\forall n \in \omega \ Y(n) \notin msl(FM(\mathbb{N})^\gamma)$$

Proof.

Suppose $\exists n Y(n) \in msl(FM(\mathbb{N})^\gamma)$.

$\exists l \forall k \geq l \ N k | = m Y(n)$

Pick a witness. $\forall k \geq l \forall x (x > n \to \neg Tr(Y(x)))$.

$\forall p \geq l \forall a < p \ N p | = m \forall x > a \ Tr(Y(x))$.

$\forall p \geq l \forall a \in (n, p) \ N p | = \neg Tr(Y(a)) \forall x > a \ Tr(Y(x))$.

$\exists q \geq p \exists b < q \ N q | = m \forall x > b \ Tr(Y(x))$.

Contradiction.
YS & the modal interpretation

Theorem

If $YD \subseteq msl(FM(\mathbb{N})^Y)$, then:

$$\forall n \in \omega \ Y(n) \notin msl(FM(\mathbb{N})^Y)$$

Proof.

- Suppose $\exists n \ Y(n) \in msl(FM(\mathbb{N})^Y)$.
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If $YD \subseteq msl(FM(N)^Y)$, then:

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Proof.

- Suppose $\exists n \ Y(n) \in msl(FM(N)^Y)$.
- $\exists l \ \forall k \geq l \ \mathbb{N}_k \models_m Y(n)$
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- Suppose $\exists n \ Y(n) \in msl(FM(\mathbb{N})^Y)$.
- $\exists l \ \forall k \geq l \ \mathbb{N}_k \models_m Y(n)$
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Proof.

- Suppose $\exists n \ Y(n) \in msl\left( FM(\mathbb{N})^Y \right)$.
- $\exists l \ \forall k \geq l \ \mathbb{N}_k \models_m Y(n)$
- Pick a witness. $\forall k \geq l \ \mathbb{N}_k \models_m \forall x (x > n \rightarrow \neg Tr(Y(x)))$.
- $\forall k \geq l \ \forall p \geq k \forall a < p \ \mathbb{N}_p \models_m a > n \rightarrow \neg Tr(Y(a))$
  - $\forall p \geq l \ \forall a \in (n, p) \ \mathbb{N}_p \models \neg Tr(Y(a))$
YS & the modal interpretation

Theorem

If \( YD \subseteq msl(\text{FM}(\mathbb{N})^\gamma) \), then:

\[ \forall n \in \omega \ Y(n) \notin msl(\text{FM}(\mathbb{N})^\gamma) \]

Proof.

\begin{itemize}
  \item Suppose \( \exists n \ Y(n) \in msl(\text{FM}(\mathbb{N})^\gamma) \).
  \item \( \exists l \ \forall k \geq l \ \mathbb{N}_k \models_m Y(n) \)
  \item Pick a witness. \( \forall k \geq l \ \mathbb{N}_k \models_m \forall x \ (x > n \rightarrow \neg Tr(Y(x))) \).
  \item \( \forall k \geq l \ \forall p \geq k \ \forall a < p \ \mathbb{N}_p \models_m a > n \rightarrow \neg Tr(Y(a)) \).
    \[ \forall p \geq l \ \forall a \in (n, p) \ \mathbb{N}_p \models \neg Tr(Y(a)) \]
  \item \( YD: \forall p \geq l \ \forall a \in (n, p) \ \mathbb{N}_p \models_m \neg Y(a) \)
\end{itemize}
YS & the modal interpretation

Theorem

If $YD \subseteq msl(FM(\mathbb{N})^Y)$, then:

$$\forall n \in \omega \ Y(n) \notin msl(FM(\mathbb{N})^Y)$$

Proof.

- Suppose $\exists n \ Y(n) \in msl(FM(\mathbb{N})^Y)$.
- $\exists l \ \forall k \geq l \ \mathbb{N}_k \models_m Y(n)$
- Pick a witness. $\forall k \geq l \ \mathbb{N}_k \models_m \forall x(x > n \rightarrow \neg Tr(Y(x)))$.
- $\forall k \geq l \ \forall p \geq k \ \forall a < p \ \mathbb{N}_p \models_m a > n \rightarrow \neg Tr(Y(a))$.
- $\forall p \geq l \ \forall a \in (n, p) \ \mathbb{N}_p \models \neg Tr(Y(a))$
- $YD: \forall p \geq l \ \forall a \in (n, p) \ \mathbb{N}_p \models_m \neg Y(a)$
- $\mathbb{N}_p \models_m \exists x > a \ Tr(Y(x))$ (content of $\neg Y(a)$)
YS & the modal interpretation

Theorem

If \( YD \subseteq \text{msl}(FM(\mathbb{N})^Y) \), then:

\[
\forall n \in \omega \ Y(n) \notin \text{msl}(FM(\mathbb{N})^Y)
\]

Proof.

- Suppose \( \exists n \ Y(n) \in \text{msl}(FM(\mathbb{N})^Y) \).
- \( \exists l \ \forall k \geq l \ \mathbb{N}_k \models_m Y(n) \)
- Pick a witness. \( \forall k \geq l \ \mathbb{N}_k \models_m \forall x (x > n \rightarrow \neg \text{Tr}(Y(x))) \).
- \( \forall k \geq l \ \forall r \geq k \forall a < p \ \mathbb{N}_p \models_m a > n \rightarrow \neg \text{Tr}(Y(a)) \)
  \[
  \forall p \geq l \ \forall a \in (n, p) \mathbb{N}_p \models_m \neg \text{Tr}(Y(a))
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- \( YD: \forall p \geq l \ \forall a \in (n, p) \mathbb{N}_p \models_m \neg Y(a) \)
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- \( \exists q \geq p \exists b < q \mathbb{N}_q \models_m b > a \land \text{Tr}(Y(b)) \)
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- \( \exists q \geq p \exists b \in (a, q) \mathbb{N}_q \models_m Tr(Y(b)) \) Contradiction.
Theorem

There is no $FM(\mathbb{N})^Y$-domain such that $YD \subseteq msl(FM(\mathbb{N})^Y)$.
No LYD

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There is no $FM(\mathbb{N})^Y$-domain such that $YD \subseteq msl(FM(\mathbb{N})^Y)$.

Proof.

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- $\forall n \forall l \exists p \geq l \exists a > n \mathbb{N}_p \models_m Tr(Y(a))$ (content of YS)
- $YD: \forall n \forall l \exists p \geq l \exists a > n \mathbb{N}_p \models_m Y(a)$.
- Let $n = l = 0$: $\exists p, a > 0 \forall q \geq p \mathbb{N}_q \models_m \forall x > a \neg Tr(Y(x))$
- Pick witness $a > 0$. $Y(a) \in msl(FM(\mathbb{N})^Y)$. Contradiction!
Summing up

Standard setting

LAD and LYD are consistent, yet $\omega$-inconsistent. Adding $\omega$-rule or UYD gives inconsistency.

$sl$-semantics YS are all false, the $sl$-theory is consistent, but $\omega$-inconsistent. Also, $sl(FM(N))$ itself is $\omega$-inconsistent.

$mm$-semantics Arithmetic regained, adding LAD and LYD gives inconsistency. UYD or $\omega$-rule are not needed.
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Finite model *in concreto*

A finite sequence of finite books each saying that all the ones behind it are false. The last one is right.
(Or so we like to think.)
Literature


