Ordinal analysis of Kripke-Platek set theory via Schmerl formula

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Schmerl formula

The arithmetical $\Pi^0_n$ uniform reflection for theory $T$

$$RFN_{\Pi^0_n}(T) : (\forall \varphi \in \Pi^0_n)(\Prv_T(\varphi) \rightarrow Tr_{\Pi^0_n}(\varphi)).$$

For recursive ordinals $\alpha$ we define r.e. theories $RFN^\alpha_{\Pi^0_n}(T)$:

$$RFN^\alpha_{\Pi^0_n}(T) = T + \{ RFN_{\Pi^0_n}(RFN^\beta_{\Pi^0_n}(T)) \mid \beta < \alpha \}.$$ 

Formally definition is carried out using Fixed Point Lemma.

EA is a weak fragment of PA proving totality of exponentiation.

Schmerl formula:

$$RFN^\alpha_{\Pi^0_{n+1}}(EA) \equiv \Pi^0_n RFN^\alpha_{\Pi^0_n}(EA), \text{ for } \alpha > 0.$$
Classifying $\Pi^0_2$ consequences of PA in terms of iterated reflection

Ordinal $\omega_n = \underbrace{\omega \cdots \omega}_{n \text{ times}}$.

\[
\begin{align*}
\text{PA} & \equiv \bigcup_{n \in \mathbb{N}} \text{RFN}_{\Pi^0_n}(\text{EA}). \\
\downarrow & \\
\text{PA} & \equiv_{\Pi^0_2} \bigcup_{n \in \mathbb{N}} \text{RFN}^{\omega_n}_{\Pi^0_2}(\text{EA}). \\
\downarrow & \\
\text{PA} & \equiv_{\Pi^0_2} \text{RFN}^{\varepsilon_0}_{\Pi^0_2}(\text{EA}).
\end{align*}
\]
From reflection to fast-growing functions

$f_\alpha(x)$ is $\alpha$’th function from fast-growing hierarchy

For any $\Delta^0_0$ formula $\varphi(x, y)$:

$$\text{RFN}^\alpha_{\Pi^0_2}(EA) \vdash \forall x \exists y \varphi(x, y)$$

\[\Downarrow\]

$$\text{RFN}^\alpha_{\Pi^0_2}(EA) \vdash \forall x (\exists y < f_2^{n+\beta}(x)) \varphi(x, y), \text{ for some } \beta < \alpha \text{ and } n \in \mathbb{N}$$

Hence

$$\text{PA} \vdash \forall x \exists y \varphi(x, y)$$

\[\Downarrow\]

$$\text{PA} \vdash \forall x (\exists y < f_\alpha(x)) \varphi(x, y), \text{ for some } \alpha < \varepsilon_0$$
**KPω vs PA**

Axioms of KP are: Extensionality, Pair, Union, $\Delta_0$-Separation, $\Delta_0$-Collection, and Foundation. 
KPω is KP + Infinity.

Transitive models of KP are known as admissible sets. 
Analogies between PA and KPω:

<table>
<thead>
<tr>
<th>PA</th>
<th>KPω</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{N}$</td>
<td>admissible sets with $\omega$</td>
</tr>
<tr>
<td>r.e. sets</td>
<td>$\Sigma_1$ classes</td>
</tr>
<tr>
<td>recursive functions</td>
<td>$\Sigma_1$ functions</td>
</tr>
<tr>
<td>recursive ordinal notations</td>
<td>$\Delta_0$ class well-orderings</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$\text{On}$</td>
</tr>
<tr>
<td>$\varepsilon_0$</td>
<td>$\varepsilon_{\text{On}+1}$</td>
</tr>
<tr>
<td>hierarchies of recursive functions</td>
<td>hierarchies of $\Sigma_1$ functions $\text{On} \rightarrow \text{On}$</td>
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<tr>
<td>r.e. theories</td>
<td>class-theories with $\Sigma_1$ class of axioms</td>
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</tbody>
</table>
Reflection principles in KP

The axioms of KP₀ are: Extensionality, Pair, Union, $\Delta_0$-Separation, $\Delta_0$-Collection, Regularity, Transitive Containment, and Totality of Rank Function.

Definitions inside KP₀:

$\mathcal{L}$ is usual set-theoretic language with constants $c_s$, for all sets $s$

*Note: $\mathcal{L}$ forms a proper class*

$\Pi_n, \Sigma_n, \Delta_0$ are $\Pi_n, \Sigma_n, \Delta_0$ with set constants.

Let $T$ be $\mathcal{L}$ theory given by $\Sigma_1$ formula defining its class of axioms.

$$RFN_{\Pi_n}(KP_0\omega) : (\forall \phi \in \Pi_n)(Prv_T(\phi) \rightarrow Tr_{\Pi_n}(\phi)).$$

$a$ ranges over $\Delta_0$ class well-orderings.

Schmerl formula:

$$RFN^a_{\Pi_{n+1}}(KP_0\omega) \equiv_{\Pi_n} RFN^{\omega a}_{\Pi_n}(KP_0\omega), \text{ for } a > 0.$$
Reformulating $\text{KP}\omega$ in terms of iterated reflection

The class well-ordering $\omega^a_n = \underbrace{\omega \cdots \omega}_{n \text{ times}}$

Over $\text{KP}_0\omega$ Foundation is equivalent to $\text{On} + 1$-iterated reflection:

\[
\text{KP}\omega \equiv \bigcup_{n \in \mathbb{N}} \text{RFN}_{\Pi_n}^{\text{On}+1}(\text{KP}_0\omega).
\]

\[
\Downarrow
\]

\[
\text{KP}\omega \equiv \Pi_2 \bigcup_{n \in \mathbb{N}} \text{RFN}_{\Pi_2}^{\omega^a_n \text{On}+1}(\text{KP}_0\omega).
\]

\[
\Downarrow
\]

\[
\text{KP}\omega \equiv \text{RFN}_{\Pi_2}^{\varepsilon^\text{On}+1}(\text{KP}_0\omega).
\]
Hierarchies of ordinal function

We have assignment of fundamental sequences $a[\xi]$, for $a < \varepsilon_{On+1}$.

$$a = \sup_{\xi < \tau_a} a[\xi], \text{ where } \tau_a \leq \text{On}$$

Bachmann defined extension of Veblen hierarchy $\varphi_a$. We use similar hierarchy $F_a$ that is closely connected to fast-growing hierarchy:

<table>
<thead>
<tr>
<th>$f_\alpha : \mathbb{N} \to \mathbb{N}$</th>
<th>$F_a : \text{On} \to \text{On}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0(n) = n + 1$</td>
<td>$F_0(\alpha) = \alpha + 1$</td>
</tr>
<tr>
<td>$f_{\alpha+1}(n) = f_\alpha(n)$</td>
<td>$F_{\alpha+1}(\alpha) = \sup_{n&lt;\omega} F_\alpha^n(\alpha)$</td>
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<tr>
<td></td>
<td>$F_a(\alpha) = \sup_{\xi &lt; \tau_a} F_a<a href="%5Calpha">\xi</a>$ if $\tau_a &lt; \text{On}$</td>
</tr>
<tr>
<td>$f_\lambda(n) = f_{\lambda[n]}(n)$</td>
<td>$F_a(\alpha) = F_{\alpha}<a href="%5Calpha">\alpha</a>$ if $\tau_a = \text{On}$</td>
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</tbody>
</table>
Ordinal bounds for $\Pi_2$ theorems of $\text{KP}_\omega$

Recall that

$$\text{KP}_\omega \equiv_{\Pi_2} \text{RFN}_{\Pi_2}^{\varepsilon_{\text{On}}+1}(\text{KP}_0\omega).$$

$$\text{RFN}_{\Pi_2}^{\alpha}(\text{KP}_0\omega) \vdash \text{"F}_b \text{ is total"}, \text{ for } b < 1 + \alpha$$

For any $\Delta_0$ formula $\varphi(x, y)$

$$\text{RFN}_{\Pi_2}^{\alpha}(\text{KP}_0\omega) \vdash \forall x \exists y \varphi(x, y)$$

$$\Downarrow$$

$$\text{RFN}_{\Pi_2}^{\alpha}(\text{KP}_0\omega) \vdash \forall x \exists y (\text{rk}(y) \leq \text{F}_{1+b}^n(\text{rk}(x)) \land \varphi(x, y))$$

and

$$\text{RFN}_{\Pi_2}^{\alpha}(\text{KP}_0\omega) \vdash \forall x \exists y (y \in L_{\text{F}_{1+b}^n(\text{rk}(x))}(x) \land \varphi(x, y))$$

for some $b < \alpha$ and $n \in \mathbb{N}$
Thank You!