Proof Theory in Philosophy

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Outline

Basics
Paradox(es)
Consistency via cut-elimination
Objects of Truth

Systems of Truth
Deflation and Conservation
Classical v Nonclassical Kripkean truth
Logical Pluralism

Extensions
Reflection
Modal Logic
Modal Predicates
What I’m **not** considering

I don’t consider *proof-theoretic semantics*.

I only briefly touch upon *reductive proof-theory* in the philosophy of mathematics.
Basics
Paradox(es)
Consistency via cut-elimination
Objects of Truth

Systems of Truth
Deflation and Conservation
Classical v Nonclassical Kripkean truth
Logical Pluralism

Extensions
Reflection
Modal Logic
Modal Predicates
Basics

Paradox(es)
Consistency via cut-elimination
Objects of Truth

Systems of Truth
Deflation and Conservation
Classical v Nonclassical Kripkean truth
Logical Pluralism

Extensions
Reflection
Modal Logic
Modal Predicates

References
Abstraction and Truth

‘There never were any set-theoretic paradoxes, but the property theoretic paradoxes are still unresolved’ (Gödel to Myhill)

Naïve abstraction

$$\forall x (x \in \{v \mid \varphi(v)\} \leftrightarrow \varphi(x))$$

Naïve Truth

$$\text{Tr} \left\langle A \right\rangle \leftrightarrow A$$

Here I assume that for any \( \varphi \) in the language there is a term \( \{v \mid \varphi(v)\} \) with \( \text{FV}(\{v \mid \varphi(v)\}) = \text{FV}(\varphi) \setminus \{v\} \). If \( \varphi \) is a sentence, I write \( \left\langle A \right\rangle \) for ‘the proposition expressed by \( A \)’.
Liar

\[ \Gamma, \Delta, \Theta, \Lambda, \ldots \text{ are multisets of formulas.} \]

Truth rules

\[
\begin{align*}
\Gamma \Rightarrow A & \quad \Gamma \Rightarrow \text{Tr} \, ^{\prime}A^{\prime} \\
A, \Gamma \Rightarrow D & \quad \text{Tr} \, ^{\prime}A^{\prime}, \Gamma \Rightarrow D \\
\lambda \iff -\text{Tr} \, ^{\prime}\lambda^{\prime} & \quad -\lambda \iff \text{Tr} \, ^{\prime}\lambda^{\prime} \\
\lambda \Rightarrow \lambda & \quad \lambda \Rightarrow \text{Tr} \, ^{\prime}\lambda^{\prime} \\
\text{Tr} \, ^{\prime}\lambda^{\prime} \Rightarrow & \quad \lambda \Rightarrow \text{Tr} \, ^{\prime}\lambda^{\prime} \\
\text{Tr} \, ^{\prime}\lambda^{\prime} & \quad \lambda, \text{Tr} \, ^{\prime}\lambda^{\prime} \Rightarrow \\
\Rightarrow -\text{Tr} \, ^{\prime}\lambda^{\prime} & \quad \lambda, \lambda \Rightarrow \\
\Rightarrow \lambda & \quad \lambda \Rightarrow \\
\Rightarrow & \\
\end{align*}
\]
Liar

Truth rules

\[
\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr} \, \langle A \rangle} \quad \frac{A, \Gamma \Rightarrow D}{\text{Tr} \, \langle A \rangle, \Gamma \Rightarrow D}
\]

\[
\lambda \iff \neg \text{Tr} \, \langle \lambda \rangle \quad \neg \lambda \iff \text{Tr} \, \langle \lambda \rangle
\]

\[
\begin{align*}
\frac{\lambda \Rightarrow \lambda}{\text{Tr} \, \langle \lambda \rangle \Rightarrow \lambda} & \quad \frac{\lambda \Rightarrow \lambda}{\text{Tr} \, \langle \lambda \rangle \Rightarrow \lambda} \\
\frac{\text{Tr} \, \langle \lambda \rangle, \neg \lambda \Rightarrow}{\text{Tr} \, \langle \lambda \rangle \Rightarrow} & \quad \frac{\lambda \Rightarrow \lambda}{\text{Tr} \, \langle \lambda \rangle \Rightarrow} \\
\frac{\Rightarrow \neg \text{Tr} \, \langle \lambda \rangle}{\Rightarrow \lambda} & \quad \frac{\lambda, \lambda \Rightarrow}{\lambda \Rightarrow}
\end{align*}
\]
Curry

Truth rules

\[
\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \text{Tr } 'A'}
\]

\[
\frac{A, \Gamma \Rightarrow D}{\text{Tr } 'A', \Gamma \Rightarrow D}
\]

\[
\kappa \iff \text{Tr } '\kappa' \rightarrow \bot
\]

\[
\frac{\kappa \Rightarrow \kappa}{\kappa \Rightarrow \text{Tr } '\kappa' \quad \bot \Rightarrow \bot}
\]

\[
\frac{\kappa, \text{Tr } '\kappa' \rightarrow \bot \Rightarrow \bot}{\kappa \Rightarrow \bot}
\]

\[
\frac{\kappa \Rightarrow \bot}{\text{Tr } '\kappa', \bot \Rightarrow \bot}
\]

\[
\frac{\kappa \Rightarrow \bot}{\kappa \Rightarrow \bot}
\]
Truth rules

\[
\begin{align*}
\Gamma \Rightarrow A & \quad \Rightarrow \quad \Gamma \Rightarrow \text{Tr} \neg \neg A^\perp \\
A, \Gamma \Rightarrow D & \quad \Rightarrow \quad \text{Tr} \neg \neg A^\perp, \Gamma \Rightarrow D \\
\kappa & \Leftrightarrow \text{Tr} \neg \neg \kappa^\perp \Rightarrow \bot \\
\kappa \Rightarrow \kappa & \quad \Rightarrow \quad \kappa \Rightarrow \text{Tr} \neg \neg \kappa^\perp \Rightarrow \bot \\
\kappa \Rightarrow \bot & \quad \Rightarrow \quad \kappa, \text{Tr} \neg \neg \kappa^\perp \Rightarrow \bot \Rightarrow \bot \\
\kappa, \text{Tr} \neg \neg \kappa^\perp \Rightarrow \bot & \quad \Rightarrow \quad \kappa \Rightarrow \bot \\
\kappa \Rightarrow \bot & \quad \Rightarrow \quad \kappa \Rightarrow \text{Tr} \neg \neg \kappa^\perp \Rightarrow \bot \Rightarrow \bot \\
\kappa, \text{Tr} \neg \neg \kappa^\perp \Rightarrow \bot & \quad \Rightarrow \quad \kappa \Rightarrow \bot \Rightarrow \bot \\
\Rightarrow \kappa & \quad \Rightarrow \quad \Rightarrow \kappa \Rightarrow \bot \\
\Rightarrow \bot & \quad \Rightarrow \quad \Rightarrow \bot \Rightarrow \bot \\
\Rightarrow \bot & \quad \Rightarrow \quad \Rightarrow \bot \Rightarrow \bot
\end{align*}
\]
Internal Curry

‘Consequence’ predicate

\[
\frac{\Gamma, A \Rightarrow B \quad \Gamma, C \Rightarrow D}{\Gamma, A, C(\neg B, \neg C) \Rightarrow D}
\]

\[
\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow C(\neg A, \neg B)}
\]

\[
\nu \Leftrightarrow C(\neg \nu, \bot)
\]

\[
\frac{\nu \Rightarrow \nu \quad \bot \Rightarrow \bot}{\nu, C(\neg \nu, \bot) \Rightarrow \bot}
\]

\[
\frac{\nu \Rightarrow \bot}{\Rightarrow C(\neg \nu, \bot)}
\]

\[
\Rightarrow \nu
\]

\[
\Rightarrow \bot
\]
Internal Curry

‘Consequence’ predicate

\[
\frac{\Gamma, A \Rightarrow B \quad \Gamma, C \Rightarrow D}{\Gamma, A, C('B', 'C') \Rightarrow D}
\]

\[
\Gamma \Rightarrow C('A', 'B')
\]

\[
\nu \iff C('\nu', \bot)
\]

\[
\frac{\nu \Rightarrow \nu \quad \bot \Rightarrow \bot}{\nu, C('\nu', \bot) \Rightarrow \bot}
\]

\[
\Rightarrow C('\nu', \bot)
\]

\[
\Rightarrow \nu
\]

\[
\Rightarrow \bot
\]

\[
\nu \Rightarrow \bot
\]

\[
\Rightarrow \bot
\]
Basics

Paradox(es)
Consistency via cut-elimination
Objects of Truth

Systems of Truth

Deflation and Conservation
Classical v Nonclassical Kripkean truth
Logical Pluralism

Extensions

Reflection
Modal Logic
Modal Predicates

References
Cut-elimination for truth and abstraction

The main extension of the standard inductive strategy consists in the reduction of cuts of the following form:

\[
\frac{\text{\textit{Tr} - rules principal in the last inferences}}{D_0 \quad D_1}
\]

\[
\begin{align*}
D_0 &: \\ 
\Gamma \Rightarrow \Delta, A &\quad \Rightarrow \Delta, \text{Tr} \; \langle A \rangle \\
\Gamma \Rightarrow \Delta, \text{Tr} \; \langle A \rangle &\quad \Rightarrow \Delta \\
\Gamma \Rightarrow \Delta &
\end{align*}
\]

\[
\begin{align*}
D_1 &: \\ 
A, \Gamma \Rightarrow \Delta &\quad \Rightarrow \Delta \\
\text{Tr} \; \langle A \rangle, \Gamma \Rightarrow \Delta &\quad \Rightarrow \Delta \\
\Gamma \Rightarrow \Delta &
\end{align*}
\]

\[
\begin{align*}
D_0 &\quad D_1
\end{align*}
\]

\[
\begin{align*}
\Gamma \Rightarrow \Delta, A &\quad \Rightarrow \Delta \\
\Gamma \Rightarrow \Delta &
\end{align*}
\]

\[
\begin{align*}
A, \Gamma \Rightarrow \Delta &\quad \Rightarrow \Delta \\
\text{Tr} \; \langle A \rangle, \Gamma \Rightarrow \Delta &\quad \Rightarrow \Delta \\
\Gamma \Rightarrow \Delta &
\end{align*}
\]

\[
\begin{align*}
\text{\textit{T}_h \text{r}} &\quad \text{\textit{T}_h}
\end{align*}
\]

This creates a problem because \text{\textit{T}_h \text{r}} \; \langle A \rangle is atomic whereas \text{\textit{T}_h} is of arbitrary (logical) complexity.
Tr -measures

- I will consider two ways of keeping track of applications of the truth rules in derivations: the first applies to nodes in the derivation tree, the second applies to single formulas within derivations.

- In the first case:

\[
\begin{align*}
\gamma_0 \Rightarrow \top \quad &\alpha \\
\gamma_0 \Rightarrow \text{Tr} \quad &\top \quad \alpha + 1 \\
\gamma_1 \Rightarrow \text{Tr} \quad &\top \quad \beta \\
\gamma_0, \gamma_1 \Rightarrow \text{Tr} \quad &\top \quad \land \quad \text{Tr} \quad \top \quad \text{max} \quad (\alpha, \beta)
\end{align*}
\]

- In the second case:

\[
\begin{align*}
\gamma_0 \Rightarrow \top ^{0} \\
\gamma_0 \Rightarrow \text{Tr} \quad \top ^{1} \\
\gamma_1 \Rightarrow \top ^{0} \\
\gamma_0, \gamma_1 \Rightarrow \text{Tr} \quad \top ^{\max(1,n)} \quad \land \quad \text{Tr} \quad \top ^{1}
\end{align*}
\]
Contraction-Free

Systems of truth and ‘set theories’ can be proved to be consistent via cut elimination arguments Grišin (1982), Petersen (2000), Cantini (2003).

**Truth à la Grišin GT**

\[
\begin{align*}
\Gamma, \text{Tr} s &\Rightarrow \text{Tr} s, \Delta [0] & \Gamma &\Rightarrow \top, \Delta [0] & \Gamma, \bot &\Rightarrow \Delta [0] \\
A, \Gamma &\Rightarrow \Delta [\alpha] & \Gamma &\Rightarrow \Delta, A [\alpha] \\
\text{Tr} \neg A', \Gamma &\Rightarrow \Delta [\alpha + 1] & \Gamma &\Rightarrow \Delta, \text{Tr} \neg A' [\alpha + 1] \\
\Gamma &\Rightarrow \Delta, A_i [\alpha] & \Gamma &\Rightarrow \Delta, A [\alpha] \\
\Gamma &\Rightarrow \Delta, A_o \cap A_1 [\alpha] & \Gamma &\Rightarrow \Delta, A \cap B [\max(\alpha, \beta)] \\
\Gamma &\Rightarrow \Delta, A [\alpha] & \Theta &\Rightarrow \Lambda, B [\beta] \\
A, B, \Gamma &\Rightarrow \Delta [\alpha] & \Gamma, \Theta &\Rightarrow \Delta, \Lambda, A \star B [\alpha + \beta] \\
A \star B, \Gamma &\Rightarrow \Delta [\alpha] & \Gamma &\Rightarrow \Delta, A [\alpha] \\
\end{align*}
\]
Systems of truth and ‘set theories’ can be proved to be consistent via cut elimination arguments Grišin (1982), Petersen (2000), Cantini (2003).

**Lemma**

*Given cut-free derivations \( D_o \vdash_{GT} \Gamma \Rightarrow \Delta, A \) and \( D_1 \vdash_{GT} A, \Theta \Rightarrow \Lambda, \) there is a \( D \vdash_{GT} \Gamma, \Theta \Rightarrow \Delta, \Lambda \) with the \( Tr \)-rank \( \rho \) of \( D \) is \( \leq \rho(D_o) + \rho(D_1). \)**

**Proof Idea.**
The induction is on \((\rho(D_o) + \rho(D_1), |A|, |D_o| + |D_1|).\)
Contraction-free

Systems of truth and ‘set theories’ can be proved to be consistent via cut elimination arguments Grišin (1982), Petersen (2000), Cantini (2003).

Two problems of the contraction-free approach:

- Viewed as a set theory, GS is inconsistent with extensionality, e.g. defined as:

\[ s \subseteq t \quad \ast \quad t \subseteq s, \ t \in r \implies s \in r \]

This is often called **Grišin’s paradox**.

- Viewed as a property theory or a truth theory, there is no known, **plausible semantics**.

However, it needs to be added that it also features a ‘decent’ conditional (compared, e.g. to Field (2008)).
Fixed-Point Semantics

Given our language $\mathcal{L}_{\text{Tr}} := \mathcal{L} \cup \{\text{Tr} \}$, we start with a (classical) model $\mathcal{M}$ of $\mathcal{L}$ such that $^\mathcal{M} \varphi = \varphi$, and set, for $X \subseteq |\mathcal{M}|$:

$$a \in \Phi(X) \iff a = \uparrow, \text{ or }$$

$$a = \text{Tr}^a \varphi \text{ and } \varphi \in X, \text{ or }$$

$$a = \lnot \text{Tr}^a \varphi \text{ and } \lnot \varphi \in X, \text{ or }$$

$$a = \varphi \land \psi \text{ and } \varphi \in X \text{ and } \psi \in X, \text{ or }$$

$$a = \lnot (\varphi \land \psi) \text{ and } \lnot \varphi \in X \text{ or } \lnot \psi \in X, \text{ or } \ldots$$

Let then $\Phi^0(X) = X, \Phi^{\alpha+1}(X) = \Phi(\Phi^\alpha(X)), \Phi^\lambda(X) = \bigcup_{\beta<\lambda} \Phi^\beta(X)$.

**Lemma (Kripke (1975), Martin and Woodruff (1975))**

If $S \subseteq |\mathcal{M}|$ is a fixed-point of $\Phi$, then for all $\varphi \in \mathcal{L}_{\text{Tr}}$:

$$\varphi \in S \iff \text{Tr}^a \varphi \in S$$
The structure \((\mathcal{M}, \mathcal{I}_\Phi)\) gives rise to a three-valued model for \(\mathcal{L}_{\text{Tr}}\) with \(\text{Tr}\) a ‘partial’ predicate. Define

\[
\mathcal{M} \models \Gamma \Rightarrow \Delta : \iff (\forall \gamma \in \Gamma) \ |\gamma|_{\mathcal{I}_\Phi} \neq \Diamond \rightarrow (\exists \delta \in \Delta) \ |\gamma|_{\mathcal{I}_\Phi} = 1
\]
Restricting initial sequents

Already known in other contexts Kreuger (1994); Jäger and Stärk (1998); Schroeder-Heister (2016). This is contained in Nicolai (2018a). Structural rules are absorbed.

Definition (LPT)

\[
\begin{align*}
\Gamma, \bot &\Rightarrow \Delta \\
\Gamma &\Rightarrow \Delta, \top, \Delta \\
\Gamma &\Rightarrow \Delta, A \\
\Gamma &\Rightarrow \Delta, \text{Tr} \ 'A' \\
\neg \bot &\Rightarrow \Delta, \phi \\
\neg \phi, \Gamma &\Rightarrow \Delta \\
\Gamma &\Rightarrow \Delta, \neg \phi \\
\vdots & \\
\vdots &
\end{align*}
\]

- Now \((\mathcal{M}, S) = \text{LPT}\) for \(S\) a fixed point of \(\Phi\).
- The model \((\mathcal{M}, \mathcal{I}_\Phi)\) satisfies a fully operational, paracomplete version system of naïve truth based on Strong-Kleene logic (modulo definition of consequence).
Back to cut elimination

When contraction is around, the notion of $\text{Tr}$-rank is not enough:

$$\begin{align*}
D_{oo} & \\
\Gamma \Rightarrow \Delta, \text{Tr} \psi', \text{Tr} \psi' [\alpha] & D_1 \\
\Gamma \Rightarrow \Delta, \text{Tr} \psi' [\alpha] & \text{Tr} \psi', \Theta \Rightarrow \Lambda [\beta] \\
\Gamma, \Theta \Rightarrow \Delta, \Lambda [\alpha + \beta]
\end{align*}$$

Now the idea here would be that we transform the derivation in

$$\begin{align*}
D_{oo}^* & \\
\Gamma \Rightarrow \Delta, \psi, \psi [\alpha] & D_1^* \\
\psi, \Theta \Rightarrow \Lambda [\beta] & D_1^* \\
\Gamma, \Theta \Rightarrow \Delta, \Lambda, \psi [\alpha + \beta] & \psi, \Theta \Rightarrow \Lambda [\beta] \\
\Gamma, \Theta, \Theta \Rightarrow \Delta, \Lambda, \Lambda [2\alpha + \beta] & \\
\Gamma, \Theta \Rightarrow \Delta, \Lambda [2\alpha + \beta]
\end{align*}$$
**Tr**-complexity $\kappa(\cdot)$ of formulas

The ordinal $\text{Tr}$-complexity $\kappa_D(\cdot)$ of a formula $\varphi$ of $\mathcal{L}_{\text{Tr}}$ in a derivation $D$ is defined inductively as follows:

- formulas of $\mathcal{L}$ **have $\text{Tr}$-complexity 0** in any $D$;
- If $D$ is just $\Gamma, \varphi \Rightarrow \varphi, \Delta$ with $\varphi \in \mathcal{L}$, then $\kappa_D(\psi) = \kappa_D(\varphi) = 0$ for all $\psi \in \Gamma, \Delta$. Similarly for $(\top), (\bot)$.
- If $D$ ends with

$$
\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \text{Tr} \ ^{A^\top}}
$$

then the complexity of formulas in $\Gamma, \Delta$ is unchanged and $\kappa_D(\text{Tr} \ ^{A^\top}) = \kappa_D(A) + 1$ (similarly for $(\text{Tr} \ ^{-L})$).
- If $D$ ends with

$$
\begin{align*}
\gamma_1^1, \ldots, \gamma_n^1 \Rightarrow \delta_1^1, \ldots, \delta_m^1, \varphi \\
\gamma_1^2, \ldots, \gamma_n^2 \Rightarrow \delta_1^2, \ldots, \delta_m^2, \psi
\end{align*}
$$

then

$$
\gamma_1^3, \ldots, \gamma_n^3 \Rightarrow \delta_1^3, \ldots, \delta_m^3, \varphi \land \psi
$$

then

$$
\begin{align*}
\kappa_D(\varphi \land \psi) &= \max(\kappa_D(\varphi), \kappa_D(\psi)) \\
\kappa_D(\gamma_i^3) &= \max(\kappa_D(\gamma_i^1), \kappa_D(\gamma_i^2)) \quad 1 \leq i \leq n \\
\kappa_D(\delta_j^3) &= \max(\kappa_D(\delta_j^1), \kappa_D(\delta_j^2)) \quad 1 \leq j \leq m
\end{align*}
$$
Full cut elimination

Crucially, rules of LPT are \( \kappa \)-invertible, e.g.: If \( \mathcal{D} \vdash_{\text{LPT}} \Gamma^1, \text{Tr} \neg A^1 \Rightarrow \Delta^1 \), then there is \( \mathcal{D}' \vdash_{\text{LPT}} A, \Gamma \Rightarrow \Delta \) with \( \kappa_{\mathcal{D}'}(\Gamma) \leq \kappa_{\mathcal{D}}(\Gamma^1), \kappa_{\mathcal{D}'}(\Delta) \leq \kappa_{\mathcal{D}}(\Delta^1) \), and

\[
\kappa_{\mathcal{D}'}(A) \leq \kappa_{\mathcal{D}}(\text{Tr} \neg A^1), \text{ if } \kappa_{\mathcal{D}}(\text{Tr} \neg A^1) = 0;
\]

\[
\kappa_{\mathcal{D}'}(A) < \kappa_{\mathcal{D}}(\text{Tr} \neg A^1), \text{ if } \kappa_{\mathcal{D}}(\text{Tr} \neg A^1) \neq 0.
\]

Lemma

Contraction is \( \kappa \)-admissible and length-admissible, e.g.: If \( \mathcal{D} \vdash^n_{\text{LPT}} \Gamma^1, \varphi^1, \varphi^2 \Rightarrow \Delta^1 \), then there is a \( \mathcal{D}' \vdash^n_{\text{LPT}} \Gamma, \varphi \Rightarrow \Delta \) with

\[
\kappa_{\mathcal{D}'}(\Gamma^1) \leq \kappa_{\mathcal{D}'}(\Gamma); \quad \kappa_{\mathcal{D}'}(\Delta^1) \leq \kappa_{\mathcal{D}'}(\Delta)
\]

\[
\kappa_{\mathcal{D}'}(\varphi) \leq \max(\kappa_{\mathcal{D}'}(\varphi^1), \kappa_{\mathcal{D}'}(\varphi^2)).
\]

Proposition

If \( \mathcal{D}_o \) is a cut-free proof of \( \Gamma^1 \Rightarrow \Delta^1, \varphi^1 \) in LPT, and \( \mathcal{D}_1 \) is a cut-free LPT-proof of \( \varphi^2, \Gamma^2 \Rightarrow \Delta^2 \), then there is a cut-free proof \( \mathcal{D} \) of \( \Gamma^3 \Rightarrow \Delta^3 \) with \( \kappa_{\mathcal{D}}(\Gamma^3) \leq \max(\kappa_{\mathcal{D}_o}(\Gamma^1), \kappa_{\mathcal{D}_1}(\Gamma^2)) \) and

\[
\kappa_{\mathcal{D}}(\Delta^3) \leq \max(\kappa_{\mathcal{D}_o}(\Delta^1), \kappa_{\mathcal{D}_1}(\Delta^2)).
\]
The ideal of **semantic closure** is at odds with resources that outstrip the ones available in one’s semantic theory. Cut-elimination procedures are usually formalizable in weak arithmetical systems.

When nonlogical initial sequents are around, full cut elimination is not in general available: however by eliminating cuts on formulas of the form \( \text{Tr} \upharpoonright A \), one can obtain **conservativity proofs**.

Another advantage of the approach with restricted initial sequents is that – unlike the contraction-free approaches – there are natural **infinitary systems** that arise from the logic and that give succinct presentations of \( \Pi^1_1 \)-sets.
Basics
Paradox(es)
Consistency via cut-elimination
Objects of Truth

Systems of Truth
Deflation and Conservation
Classical v Nonclassical Kripkean truth
Logical Pluralism

Extensions
Reflection
Modal Logic
Modal Predicates
Truth Bearers

- There are good reasons to require an ontology of bearers of truth prior to discussing principles of truth. We want to prove in the **object language** things like:

\[
\forall \varphi, \psi \exists \chi (\chi = (\varphi \land \psi) \land \varphi \neq \chi)
\]

This is usually guaranteed by assuming a theory of finite objects (as we shall see in a moment).

- Notice that this is imposing non-trivial constraints. More ‘philosophical’ theories of truth are often formulated in terms of **propositions**, and not sentence types (Horwich, 1998; Soames, 1998; Jago, 2018). This rules out that propositions are coarse-grained, e.g. sets of possible worlds.
Arithmetic

- Peano arithmetic (PA) is the preferred base theory for systems of truth. It is usually formulated in $\mathcal{L}_\mathbb{N} = \{0, S, +, \times\}$ and features equations for its primitives, e.g.
  \[
  (x + 0) = x \quad \quad \quad x + Sy = S(x + y)
  \]
  and the first-order induction schema
  \[
  \varphi(0) \land \forall x(\varphi(x) \rightarrow \varphi(Sx)) \rightarrow \forall x \varphi(x) \quad \text{for } \varphi \in \mathcal{L}_\mathbb{N}
  \]

- Alternatively, one can employ a theory of strings and concatenation $\wedge$ with two atoms $a, b$ based on Tarski’s axiom
  \[
  a \wedge y = u \wedge v \leftrightarrow \exists w((x = u \wedge w \land v = w \wedge y) \lor (u = x \wedge w \land y = w \wedge v))
  \]
  With first-order string induction, the two theories are mutually interpretable. An accessible source is Ganea (2009).

- Finite set theories are also a convenient choice. For instance
  - Kaye and Wong (2006) show that PA and KF \{\text{Inf}\}+ ‘every set has a transitive closure’ are bi-interpretable;
  - similarly, a neat set theory based by Świerczkowski (2003) based on the adjunction operation $x \triangleleft y \mapsto x \cup \{y\}$ is bi-interpretable with PA.
Doing with less

- Ultimately, what we require to establish the basic properties of the truth bearers are a **good notion of sequence**, and a **minimum of induction** to handle suitable forms of recursion.

- For the former the notion of a **sequential** theory is enough – see Visser (2010) for a comprehensive overview. A theory is sequential if it interprets – with no relativization of quantifiers – the theory AS given by the empty set axiom and adjunction – which is as strong as Robinson’s Q.

- As to induction, since all the relevant syntactic notions (terms, formulas, proofs) are **p-time decidable**, the theory $S^1_2$ by Buss (1986) suffices. However, many of the results that I will treat below are specific to PA (or equivalents), and it is object of current research to check which results are stable over $T \supseteq S^1_2$. 
A good base theory will also provide a satisfactory representation of ordinals. For our purposes it suffices to require a notation up to the Feferman-Schütte ordinal $\Gamma_0$:

- $\alpha$ is principal if it cannot be expressed as $\zeta + \eta$ for $\zeta, \eta < \alpha$. Define:

  \[
  C(0) := \text{`the class of principal ordinals'}
  \]

  \[
  C(\alpha + 1) := \text{`the class of fixed points of the function enumerating } C(\alpha)'
  \]

  \[
  C(\lambda) := \bigcap_{\zeta<\lambda} C(\zeta) \text{ for } \lambda \text{ a limit ordinal}
  \]

- The Veblen functions $\varphi_\alpha$ are the enumerating functions of $C(\alpha)$. The class of strongly critical ordinals SC contains precisely the ordinals $\alpha$ that are themselves $\alpha$-critical. $\Gamma_\zeta$ indicates the $\zeta$-th strongly critical ordinal.

- Principal ordinals $\alpha$ that are not themselves strongly critical are such that $\alpha = \varphi_\zeta \eta$ for $\eta, \zeta < \alpha$. Therefore, by this fact and Cantor's normal form theorem, ordinals $< \Gamma_0$ can be uniquely determined as words of the alphabet ($0, +, \varphi \cdot$).
A good base theory will also provide a satisfactory representation of ordinals. For our purposes it suffices to require a notation up to the Feferman-Schütte ordinal $\Gamma_0$.

A notation system for $\Gamma_0$ is of the form $(\mathcal{OT}, \mathcal{PT}, |\cdot|, <)$, with:

- $\mathcal{OT}$ the set of natural number ‘codes’ for ordinals $< \Gamma_0$
- $\mathcal{OT} \subseteq \mathcal{OT}$ the set of codes of principal ordinals
- $|\cdot| : \mathcal{OT} \rightarrow \mathbb{ON}$
- $n < m :\leftrightarrow n \in \mathcal{OT} \land m \in \mathcal{OT} \land |n| < |m|$

Using standard coding techniques one can show that $\mathcal{OT}$, $\mathcal{PT}$, $<$ are primitive recursive. Actually, Beckmann et al. (2003) show that they can be showed to be p-time and represented in $S^1_2$ – notice that I do not mean that $\Gamma_0$ can be well-founded in $S^1_2$!
Schemata

For ordinals $\alpha < \Gamma_0$, we denote with $a$ the corresponding numeral in the representation of $0T$ and we do not distinguish between ordinal functions such as the Veblen functions and their representations. The system $(0T, PT, <)$ enables us to formulate the following principles of transfinite induction:

$$(T^{\varepsilon_0}_{\mathcal{L}_{Tr}}) \quad \forall a < b \phi(a), \Gamma \Rightarrow \Delta, \phi(b) \quad \Gamma \Rightarrow \Delta, \forall a < \varepsilon_0 \phi(a)$$

$$(T^{\omega^\omega}_{\mathcal{L}_{Tr}}) \quad \forall a < b \phi(a), \Gamma \Rightarrow \Delta, \phi(b) \quad \Gamma \Rightarrow \Delta, \forall a < \omega^\omega \phi(a)$$

$$(T^{\varepsilon_0}_{\mathcal{L}_{Tr}}) \quad \forall a < b \phi(a), \Gamma \Rightarrow \Delta, \phi(b) \quad \Gamma \Rightarrow \Delta, \forall a < \varepsilon_0 \phi(a)$$

for all $\gamma(= |c|) < \omega^\omega$

for all $\gamma < \varepsilon_0$
Basics
- Paradox(es)
- Consistency via cut-elimination
- Objects of Truth

Systems of Truth
- Deflation and Conservation
- Classical v Nonclassical Kripkean truth
- Logical Pluralism

Extensions
- Reflection
- Modal Logic
- Modal Predicates

References
Basics
Paradox(es)
Consistency via cut-elimination
Objects of Truth

Systems of Truth
Deflation and Conservation
Classical v Nonclassical
Kripkean truth
Logical Pluralism

Extensions
Reflection
Modal Logic
Modal Predicates
Truth as primitive

- Truth-theoretic deflationism holds that truth is **not a genuine property** and that its function is mainly that of a **generalizing device** (Quine, 1970; Field, 1994; Horwich, 1998).

- Unlike other notions that have been taken to be primitive for lack of consensus over a definition – e.g. knowledge, see Williamson (2000) – **Tarski’s theorem** uncontroversially establishes this (Halbach, 2014, Ch. 1).

- Truth is a fundamental **semantic** concept. A theory of meaning for natural language expressions is not much more than a (Tarskian) **theory of truth** for it (Davidson, 1984).
Tarskian Truth

The theory of truth in $\mathcal{L}_{Tr}$ that Davidson had in mind extends PA with the following:

**Definition (CT)**

\[
\forall s, t (\text{Tr} (s = t) \iff s^o = t^o) \\
\forall \varphi \in \mathcal{L} (\text{Tr} (\neg \varphi) \iff \neg \text{Tr} \varphi) \\
\forall \varphi, \psi \in \mathcal{L} (\text{Tr} (\varphi \land \psi) \iff \text{Tr} \varphi \land \text{Tr} \psi) \\
\forall v, \forall \varphi(v) \in \mathcal{L} (\text{Tr} (\forall v \varphi) \iff \forall x \text{Tr} \varphi(\dot{x})) \\
\varphi(o) \land \forall x (\varphi(x) \rightarrow \varphi(x + 1)) \rightarrow \forall x \varphi(x) \quad \text{with } \varphi(v) \in \mathcal{L}_{Tr}
\]

Important variations are obtained by tweaking the induction schema:

- **CT** $\uparrow$ (a.k.a. **CT**$^-$) is obtained by restricting induction to $\mathcal{L}$
- **CT$\text{int}$** is obtained by adopting the internal induction schema

\[
\forall \varphi(v) (\text{Tr} \varphi(o/v) \land \forall y (\text{Tr} \varphi(y/v) \rightarrow \text{Tr} \varphi(Sy/v)) \rightarrow \forall x \text{Tr} \varphi(\dot{x}/v))
\]
Conservativeness

**Thesis (Shapiro, 1998; Ketland, 1999)**

CT proves Con(PA), therefore deflationism is untenable.

This contrasts with:

**Proposition (Kotlarski et al. (1981); Visser and Enayat (2015); Leigh (2015))**

\( \text{CT}_{int} \) is a conservative extension of PA.

The discussion took a strong technical turn, brilliantly summarized in Cieliski (2017) – with many original contributions. It’s worth mentioning:

**Proposition (Enayat and Pakhomov (2018))**

CT \( \uparrow \) plus ‘disjunctive correctness’, i.e.

\[
\forall s \left( \text{Tr} \left( \bigvee_{i<s} \varphi_i \right) \leftrightarrow \exists i < s \ \text{Tr} \varphi_i \right)
\]

is the same theory as CT[\( \text{I} \Delta_0 \)], and therefore proves Con(PA).
Conservativeness

- Despite the technical interest, the debate seems to be built on shaky foundations. Virtually no deflationist has thoroughly defended the claim that truth has to be conservative over the base theory.

- By contrast, it has repeatedly been argued that truth has to be nonconservative, but in a way that is distinctively metalinguistic, i.e. it does not interfere with the subject matter of the base theory over which truth is built.

- This has led to the programme of ‘disentangling’ syntactic quantifiers from quantifiers over natural numbers:

**Proposition (Nicolai (2015, 2016))**

*If one formulates CT $\uparrow$ as a two-sorted theory, with ‘syntactic’ quantifiers and ‘number-theoretic’ quantifiers, and truth applying only over syntactic objects, then:*

- The theory of truth becomes trivially conservative over PA;

- This version of CT $\uparrow$ plus ‘all axioms of PA are true’ is mutually interpretable with PA + Con(PA).
Basics

Paradox(es)
Consistency via cut-elimination
Objects of Truth

Systems of Truth

Deflation and Conservation
Classical v Nonclassical Kripkean truth
Logical Pluralism

Extensions

Reflection
Modal Logic
Modal Predicates

References
Feferman’s project

To give a nice presentation of the **reflective closure** of PA – and possibly of further ‘natural’ theories: i.e. the (truth-)theory that makes explicit all that is implicit in the acceptance of PA.

**First attempt: CT_{<a}, \alpha \leq \Gamma_0**

\[
\mathcal{L}_o := \mathcal{L}_{Tr} \quad \quad \mathcal{L}_{<c} := \mathcal{L} \cup \{\text{Tr}_b \mid b < c\}
\]

With \(b < c\):

\[
\forall s, t (\text{Tr}_b (s = t) \leftrightarrow s^o = t^o)
\]
\[
\forall \varphi \in \mathcal{L}_{<b} (\text{Tr}_b (\neg \varphi) \leftrightarrow \text{Tr}_b (\neg \varphi))
\]
\[
\forall \varphi, \psi \in \mathcal{L}_{<b} (\text{Tr}_b (\varphi \land \psi) \leftrightarrow \text{Tr}_b \varphi \land \text{Tr}_b \psi)
\]
\[
\forall \nu, \forall \varphi (\nu) \in \mathcal{L} (\text{Tr}_b (\forall \nu \varphi) \leftrightarrow \forall x \text{Tr}_b \varphi (\dot{x}))
\]
\[
\forall \varphi \in \mathcal{L}_{<a<b} (\text{Tr}_b \text{Tr}_a \varphi \leftrightarrow \text{Tr}_a \varphi)
\]
\[
\forall d < b, \forall \varphi \in \mathcal{L}_{<d} (\text{Tr}_b \text{Tr}_d \varphi \leftrightarrow \text{Tr}_b \varphi)
\]
Feferman’s project

- The project of isolating $\text{CT}_{<\varepsilon_0}$ or $\text{CT}_{<\Gamma_0}$ as natural stopping points, that was congenial to Feferman’s project, depended essentially on other results, such as Feferman’s and Schütte’s independent characterization of $\Gamma_0$, or the provable well-orderings of PA.

- The next step was to find an independent characterization of such theories:

**Definition (KF)**

\[
\begin{align*}
\forall s, t (\text{Tr}(s = t) & \leftrightarrow s^\circ = t^\circ) \\
\forall s, t (\text{Tr}(s \neq t) & \leftrightarrow s^\circ \neq t^\circ) \\
\forall t (\text{Tr} (t) & \leftrightarrow \text{Tr}(t^\circ)) \\
\forall t (\text{Tr} (\neg t) & \leftrightarrow \text{Tr}(\neg t^\circ)) \\
\forall \phi \in \mathcal{L}_{\text{Tr}} (\text{Tr}(\neg \phi) & \leftrightarrow \text{Tr}(\phi)) \\
\forall \phi, \psi \in \mathcal{L}_{\text{Tr}} (\text{Tr}(\phi \land \psi) & \leftrightarrow \text{Tr}(\phi \land \text{Tr}(\psi)) \\
\forall \phi, \psi \in \mathcal{L}_{\text{Tr}} (\text{Tr}(\neg(\phi \land \psi) & \leftrightarrow \text{Tr}(\neg \phi \lor \text{Tr}(\neg \psi)) \\
\forall \nu, \forall \phi(\nu) \in \mathcal{L}_{\text{Tr}} (\text{Tr}(\forall \nu \phi) & \leftrightarrow \forall x \text{Tr}(\phi(\dot{x}))) \\
\forall \nu, \forall \phi(\nu) \in \mathcal{L}_{\text{Tr}} (\text{Tr}(\neg \forall \nu \phi) & \leftrightarrow \exists x \text{Tr}(\neg \phi(\dot{x})))
\end{align*}
\]
Properties of KF

- Semantically KF fits nicely with Kripke’s fixed-point semantics (Kripke, 1975),

\[(\mathbb{N}, S) \models \text{KF} \iff S \text{ is a fixed point of Kripke’s theory of truth}\]

- The full \(\text{Tr}\) -schema is available for meaningful predicates satisfying \(\text{D}(x) :\leftrightarrow \text{Tr} x \lor \text{Tr} \neg x\), i.e. for all \(A \in \mathcal{L}_{\text{Tr}}:\)

\[\text{KF} \vdash \text{D}(\text{"}A\text{"}) \rightarrow (\text{Tr} \text{"}A\text{"} \leftrightarrow A)\]

Proposition (Feferman (1991); Cantini (1989))

KF is proof-theoretically equivalent to \((\Pi_1^0 - \text{CA})_{\varepsilon_0}\).

Proof Idea.

Lower bound: PA in \(\mathcal{L}_{\text{Tr}}\) proves \(\text{TI}_{\mathcal{L}_{\text{Tr}}}^{\varepsilon_0}\). Now KF proves:

\[\varphi \in \mathcal{L}_{<a} \rightarrow \text{D}(\varphi) \Rightarrow \varphi \in \mathcal{L}_a \rightarrow \text{D}(\varphi)\]

An application of \(\text{TI}_{\mathcal{L}_{\text{Tr}}}^{\varepsilon_0}\) yields an embedding of \(\text{CT}_{\varepsilon_0}\), which suffices.
Properties of KF

- Semantically KF fits nicely with Kripke’s fixed-point semantics (Kripke, 1975),

  \((\mathbb{N}, S) \models \text{KF} \iff S \text{ is a fixed point of Kripke’s theory of truth}\)

- The full \(\text{Tr}\) -schema is available for meaningful predicates satisfying \(D(x) :\leftrightarrow \text{Tr } x \lor \text{Tr } \neg x\), i.e. for all \(A \in \mathcal{L}_{\text{Tr}}:\)

\[
\text{KF} \vdash D(\ulcorner A \urcorner) \rightarrow (\text{Tr } \ulcorner A \urcorner \leftrightarrow A)
\]

**Proposition (Feferman (1991); Cantini (1989))**

KF is proof-theoretically equivalent to \((\Pi^0_1 - \text{CA})_{< \varepsilon_0}\).

**Proof Idea.**

**Upper bound:** Oneformulates KF in a Tait (one-sided) infinitary calculus, and analyzes quasi-normal derivations, i.e. derivations with only cuts on \(\text{Tr } t\) and \(\neg \text{Tr } t\) and proves in \(\text{CT}_{< \varepsilon_0}\) that

\[
\text{if } \text{KF}^\infty \vdash^\alpha \text{Tr } \ulcorner A \urcorner, \text{ then } \text{Tr }_2^\alpha \ulcorner A \urcorner
\]
Symmetries

- One important drawback of KF is that its internal theory \( \{ \varphi \in \mathcal{L}_{\text{Tr}} \mid \text{KF} \vdash \text{Tr} \left[ \varphi \right] \} \) is different from its theorems: for instance \( \text{KF} \vdash \lambda \lor \neg \lambda \) but \( \text{KF} \nvdash \text{Tr} \left[ \lambda \lor \neg \lambda \right] \).

- To overcome this:

Reinhardt’s thesis

One should adopt an instrumental reading of KF. Its conceptual core is given by its internal theory.

Lemma (Halbach and Horsten (2006))

There are A’s in \( \mathcal{L}_{\text{Tr}} \) such that \( \text{KF} \vdash \text{Tr} \left[ A \right] \) but the proof essentially employs B’s such that \( \text{KF} \nvdash \text{Tr} \left[ B \right] \).
Kripke-Feferman in four-valued logic

- From classical logic, **remove** both negation introduction rules are replace them with, e.g.:
  \[
  \begin{align*}
  \Gamma \Rightarrow \Delta, \neg \varphi, \neg \psi & \quad \neg \varphi, \Gamma \Rightarrow \Delta \quad \neg \psi, \Gamma \Rightarrow \Delta \\
  \Gamma \Rightarrow \Delta, \neg (\varphi \land \psi) & \quad \neg (\varphi \land \psi), \Gamma \Rightarrow \Delta
  \end{align*}
  \]
  One obtains a logic in the vicinity of **FDE**.

- As to truth:

**PKF**

\[
\begin{align*}
  s^o = t^o & \iff \text{Tr}(s = t) \\
  \neg \text{Tr} \varphi & \iff \text{Tr} \neg \varphi \\
  \text{Tr}(\varphi \land \psi) & \iff \text{Tr} \varphi \land \text{Tr} \psi \\
  \text{Tr} \text{Tr} \varphi & \iff \text{Tr} \varphi
\end{align*}
\]

- The theories are closely related:

**ω-categoricity**

For S a fixed point of Kripke’s theory of truth,

\[
(\mathbb{N}, S) \models KF \iff (\mathbb{N}, S) \models_{FDE} PKF
\]
Kripke-Feferman in four-valued logic

Unlike KF, $\text{PKF} \vdash \text{Tr} \, 'A$ iff $\text{PKF} \vdash A$, for all $A \in \mathcal{L}_{\text{Tr}}$.

**Proposition (Halbach and Horsten (2006))**

$\text{PKF}$ proves the same arithmetical sentences as $\text{CT}_{<\omega^\omega}$.

**Proposition (Feferman (1991))**

$\text{KF}$ proves the same arithmetical sentences of $\text{CT}_{<\varepsilon_0}$.

Recalling Reinhardt’s thesis:

**Corollary**

*The internal theory of KF and PKF differ considerably.*

**Proposition (Nicolai (2018b))**

- $\text{PKF} = \{ A \in \mathcal{L}_{\text{Tr}} \mid \text{KF}_{\text{int}} \vdash \text{Tr} \,'A$ \}
- $\{ A \in \mathcal{L}_{\text{Tr}} \mid \text{KF} \vdash \text{Tr} \,'A$ \} = \text{PKF} + \text{T}I_{<\varepsilon_0}$
<table>
<thead>
<tr>
<th>PKF, KF_{int}</th>
<th>\mathcal{L}</th>
<th>\mathcal{L}_{Tr}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>\varphi_{\omega} \circ \omega</td>
<td>\varphi_{\circ} \omega \ (or \ \omega^{\omega})</td>
</tr>
<tr>
<td>KF</td>
<td>\varphi_{\varepsilon_{0}} \circ \omega</td>
<td>\varphi_{1} \circ \ (or \ \varepsilon_{0})</td>
</tr>
</tbody>
</table>
Basics

Paradox(es)
Consistency via cut-elimination
Objects of Truth

Systems of Truth

Deflation and Conservation
Classical v Nonclassical Kripkean truth
Logical Pluralism

Extensions

Reflection
Modal Logic
Modal Predicates
The costs of nonclassical logic

- This asymmetry between the amount of transfinite induction provable in PKF and KF has been taken – see Halbach (2014) and Halbach and Nicolai (2018) – as substantiating Feferman’s claim that

  ‘nothing like sustained ordinary reasoning can be carried out [in such logic]’ (Feferman, 1984)

- This seems to be supported by:

  Halbach and Nicolai (2018)

  \[ \text{PKF} \uparrow \mathcal{L} = \{ \varphi \in \mathcal{L}_{\text{Tr}} \mid \text{KF} \uparrow \mathcal{L} \vdash \text{Tr} \left[ \varphi \right] \}. \]

  - A possible rejoinder is that transfinite induction open to semantic notions cannot be taken as compromising mathematical reasoning.
**INDEC** is the assertion that:

> every countable scattered indecomposable linear ordering is either indecomposable to the left, or indecomposable to the right.

It was proved to be true by Pierre Jullien in 1969.

**Lemma (Montalbán, Friedman)**

**INDEC** is implied by $\Sigma^1_1$-CA, which is $\Pi^1_2$-conservative over ACA$_{<\varepsilon_0}$.

**Proposition (Eastaugh, N.)**

**RCA**+$\text{INDEC}$ is proof-theoretically equivalent to **KF**. It follows that **KF** can ‘nicely’ interpret **RCA**+$\text{INDEC}$, but **KF** cannot.
Basics
Paradox(es)
Consistency via cut-elimination
Objects of Truth

Systems of Truth
Deflation and Conservation
Classical v Nonclassical Kripkean truth
Logical Pluralism

Extensions
Reflection
Modal Logic
Modal Predicates
Basics
Paradox(es)
Consistency via cut-elimination
Objects of Truth

Systems of Truth
Deflation and Conservation
Classical v Nonclassical Kripkean truth
Logical Pluralism

Extensions
Reflection
Modal Logic
Modal Predicates
We have seen that $\text{CT}_{<\epsilon_0}$, $\text{CT}_{<\Gamma_0}$, KF, and in part PKF can be seen as attempt to characterize the **reflective closure** of PA.

There is a more direct strategy involving **reflection principles**, i.e. claims of the form

\[
\text{RFN}(S) \quad \text{if something is provable in a theory } S, \text{ then it's true, or}
\]

\[
\forall x (\text{Prov}_S('A(x)) \rightarrow A(x))
\]

It turns out that if one focuses on **classical** biconditionals

\[
\text{PTB} \upharpoonright \mathcal{L} = \{ \text{Tr } 'A' \leftrightarrow A \mid A \text{ positive of } \mathcal{L}_{\text{Tr}} \}
\]

one obtains:

**Proposition (Horsten and Leigh (2017))**

\[
\text{RFN}^2(\text{PTB} \upharpoonright \mathcal{L}) \supseteq \text{KF}.
\]
There seems to be a conceptual problem with this strategy:

If oneformulates reflection – as it seems plausible – as:

$$\forall \varphi (\text{Prov}_{PTB}(\varphi) \rightarrow \text{Tr} \varphi)$$

then there is a sentence $A$ such that PTB with this form of reflection proves $\text{Tr} \leftarrow\begin{array}{c} A \land \neg A \end{array}$.

A possible way out is to start with a set of ‘biconditionals’ over FDE:

$$TS \upharpoonright L = \{ \text{Tr} \leftarrow\begin{array}{c} A \end{array} \leftrightarrow A \mid A \in L_{Tr} \}$$

and define reflection as

$$\begin{array}{c}
\frac{\text{Pr}_S^2 (\leftarrow\begin{array}{c} \Gamma \Rightarrow \Delta x, \Theta x \Rightarrow \Lambda x \end{array}) \Gamma(x) \Rightarrow \Delta(x)}{\Theta(x) \Rightarrow \Lambda(x)}
\end{array}
$$

Then we can prove:

**Proposition (Fischer et al. (2017))**

- $RR^2 (TS \upharpoonright L) \supseteq \text{PKF}$.
- $RR^\omega (TS \upharpoonright L) \vdash TI_{\leq \omega}^\omega \subseteq \omega^2$
Basics
Paradox(es)
Consistency via cut-elimination
Objects of Truth

Systems of Truth
Deflation and Conservation
Classical v Nonclassical Kripkean truth
Logical Pluralism

Extensions
Reflection
Modal Logic
Modal Predicates
An analogy

- Solovay’s theorem tells us that, for all $A \in \mathcal{L}_\square$, 

  $$GL \vdash A \text{ iff } \forall \star (PA \vdash A^\star)$$

- By redefining a realization $\star: \mathcal{L}_\square \rightarrow \mathcal{L}_{Tr}$ as:

  $$A^\star = \begin{cases} 
  0 = 1, & \text{if } A = \bot, \\
  \text{commutes with connectives} & \\
  \text{Tr} \ l B^\star & \text{if } A = \square B
  \end{cases}$$

  one can ask:

**Question**

$$? \vdash A \iff \forall \star (KF \vdash A^\star)$$
Nicolai and Stern (2018)

The logic $L_{\Box}$

\[
\begin{align*}
\Box \top & \\
\Box A & \leftrightarrow \Box \neg \neg A \\
\Box (A \land B) & \leftrightarrow \Box A \land \Box B \\
\Box A & \leftrightarrow \Box \Box A
\end{align*}
\]

\[
\begin{align*}
\neg \Box \bot & \\
\Box A \land \neg \Box \neg A & \rightarrow A \\
\neg \neg (A \land B) & \leftrightarrow \Box \neg A \lor \Box \neg B \\
\Box A & \leftrightarrow \Box \neg \Box A
\end{align*}
\]

Proposition

$L_{\Box} \vdash A$ iff $\forall \star (KF \vdash A^\star)$
Basics
Paradox(es)
Consistency via cut-elimination
Objects of Truth

Systems of Truth
Deflation and Conservation
Classical v Nonclassical Kripkean truth
Logical Pluralism

Extensions
Reflection
Modal Logic
Modal Predicates


