Proof mining of the proximal point algorithm with multi-parameters

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Proof mining

Proof mining program $\rightarrow$ analysis of mathematical proofs with the help of proof theoretic techniques, including functional interpretations, in search of concrete new information: effective bounds, algorithms, weakening of premisses, ...

Proof mining with the BFI

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- Completely new translation of formulas.
- Independence on bounded parameters is made explicit.

Framework

Let H be a real Hilbert space with inner product $\langle \cdot \rangle$ and norm $\|\cdot\|$ and let $T : H \to 2^H$ be an operator in H.
T is monotone if

$$(x, y), (x', y') \in \Gamma(T) \Rightarrow \langle x - x', y - y' \rangle \geq 0.$$

A monotone operator T is maximal monotone if $\Gamma(T)$ is not properly contained in the graph of any other monotone operator on H.
We denote by S the (nonempty) set of all zeros of T, i.e., $S = T^{-1}(0)$.

For $c > 0$, we use J_c to denote the resolvent of T, i.e. the single-valued function defined by

$$J_c = (I + cT)^{-1}.$$

A mapping $f : K \to K$ is called nonexpansive if for all $x, y \in K$

$$\|f(x) - f(y)\| \leq \|x - y\| .$$

The resolvent J_c is nonexpansive, and

$$\boxed{Fix(J_c) = S}$$

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- Many problems can be formulated as finding a zero of maximal monotone operators.
- PPA is a powerful and successful algorithm in finding a solution of maximal monotone operators.
- Starting from any initial guess $z_0 \in H$, the PPA generates a sequence which approximates the solution.

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- We will focus on a theorem of Yao and Noor for which we have the strong convergence of the algorithm to the nearest projection point onto the set of zeros of the operator.

mPPA

Let H be a Hilbert space. The proximal point algorithm with multi-parameters is the following algorithm

$$\mathrm{mPPA}: \boxed{z_{n+1} = \lambda_n u + \gamma_n z_n + \delta_n J_{c_n}(z_n) + e_n,\ n \geq 0,}$$

where

- $u \in H$ is given,
- $c_n > 0$,
- $\lambda_n, \gamma_n, \delta_n \in (0,1)$ and
- $\lambda_n + \gamma_n + \delta_n = 1, \forall n \geq 0.$

A theorem by Yao-Noor

$$\boxed{z_{n+1} = \lambda_n u + \gamma_n z_n + \delta_n J_{c_n}(z_n) + e_n}$$

Theorem

Let (z_n) be generated by mPPA*. Assume that*

(i) $\lim_{n\to\infty} \lambda_n = 0$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$;

(iv) $c_n \geq c$, *where c is some positive constant;*

(v) $c_{n+1} - c_n \to 0$;

(vi) $\sum_{n=1}^{\infty} \|e_n\| < \infty$.

Then (z_n) converges strongly to a point $z \in S$ which is the nearest point projection of u onto S.

Quantitative version of Yao-Noor's theorem I

Let (z_n) be generated by mPPA. Assume that there exist $a, b, c, d \in \mathbb{N}$ and $s \in S$ and monotone functions l, L, Δ, Γ, E such that

(i) $\forall k \in \mathbb{N} \forall n \geq l(k) \left(\lambda_n \leq \frac{1}{k+1} \right)$;

(ii) $\forall k \in \mathbb{N} \left(\sum_{i=1}^{L(k)} \lambda_i \geq k \right)$;

(iii) $\forall n \geq a \left(\frac{1}{a+1} \leq \gamma_n \leq 1 - \frac{1}{a+1} \right)$;

(iv) $\forall n \in \mathbb{N} \left(c_n \geq \frac{1}{c+1} \right)$;

(v) $\forall n \in \mathbb{N} \left(\frac{1}{\Delta(n)+1} \leq \eta_n \leq 1 - \frac{1}{\Delta(n)+1} \right), \eta_n \in \{\lambda_n, \gamma_n, \delta_n\}$;

(vi) $\forall k \in \mathbb{N} \forall n \geq \Gamma(k) \left(|c_{n+1} - c_n| \leq \frac{1}{k+1} \right)$;

(vii) $\forall k \in \mathbb{N} \forall n \in \mathbb{N} \left(\sum_{i=E(k)+1}^{E(k)+n} \|e_i\| \leq \frac{1}{k+1} \right)$.

Quantitative version of Yao-Noor's theorem II

Theorem

Under the assumptions (i)-(vii) *we have that*

$$\forall k \in \mathbb{N} \forall f : \mathbb{N} \to \mathbb{N} \exists n \leq \phi(k, f) \forall i, j \in [n, n+fn] \left(\|z_i - z_j\| \leq \frac{1}{k+1} \right),$$

where $\phi(k, f)$ is a finitely recursive bound explicitly given in the proof.

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- We were also able to eliminate an argument of weak compactness. For such elimination the BFI seems to be more intuitive and easy to carry out than Kohlenbach's monotone interpretation.

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- We were able to eliminate this dependency by using the fact that the sequence in question is bounded and making a rational approximation.
- We were also able to eliminate an argument of weak compactness. For such elimination the BFI seems to be more intuitive and easy to carry out than Kohlenbach's monotone interpretation.
- Moreover countable choice (in the projection argument) was eliminated due to a previous observation by Kohlenbach.

References I

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Thank you!