Provability Logic
Day 2
Connections to proof-theoretic ordinals

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Review: Arithmetical languages

We assume that $0, +, \times, 2^x$ are definable, as is quantification over $\mathbb{N}$

Δ_0: Formulae where all quantifiers are of the form $\forall x{<}t$ or $\exists x{<}t$

Π_n: $\forall x_1 \exists x_2 \ldots Q_n x_n \varphi$ with $\varphi \in \Delta_0$

Σ_n: $\exists x_1 \forall x_2 \ldots Q_n x_n \varphi$ with $\varphi \in \Delta_0$

Review: Arithmetical theories

EA: Allows induction for Δ_0 formulas

IΓ: Allows induction for formulas in Γ

PA : Induction for all formulas

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Standing assumption: Theories are computably enumerable, sound, extend EA

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$\mathtt{Prv}_T(x)$ is a Σ_1 formula that defines provability in T

Theorem (Provable Σ_1-completeness:)
If $\sigma(x) \in \Sigma_1$,

$$\mathrm{EA} \vdash \forall x \left(\sigma(x) \to \mathtt{Prv}_{\mathrm{EA}}(\ulcorner \sigma(\dot{x}) \urcorner)\right)$$

Review: Gödel-Löb logic

Language:

$$p \qquad \neg\varphi \qquad \varphi \wedge \psi \qquad \Box\varphi$$

Axioms:

- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ (Löb's axiom)

1. sound and complete for the class of finite (and thus well founded) strict partial orders
2. sound and complete for its arithmetical interpretation: $(\Box\varphi)^f = \mathtt{Prv}_T(\ulcorner\varphi^f\urcorner) \in \Sigma_1$

Soundness of Löb's axiom

Proof 1.

1. Apply the second incompleteness theorem to $T + \neg\varphi$

$$\Box_{T+\neg\varphi}\Diamond_{T+\neg\varphi}\top \rightarrow \Box_{T+\neg\varphi}\bot$$

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2. By the deduction theorem we can replace $\Box_{T+\neg\varphi}\psi$ by $\Box_T(\neg\varphi \to \psi)$

$$\Box_T(\neg\varphi \to \neg\Box_T\neg\neg\varphi) \to \Box_T\neg\neg\varphi$$

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3. Simplify

$$\Box_T(\Box_T\varphi \to \varphi) \to \Box_T\varphi$$

□

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6. $T \vdash S$

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6. $T \vdash S$
7. $T \vdash \Box_T S$
8. $T \vdash \varphi$

Review: Japaridze's polymodal logic

GLP:

$$[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi) \qquad (n < \omega)$$
$$[n]([n]\varphi \to \varphi) \to [n]\varphi \qquad (n < \omega)$$
$$[n]\varphi \to [m]\varphi \qquad (n < m < \omega)$$
$$\langle n\rangle\varphi \to [m]\langle n\rangle\varphi \qquad (n < m < \omega)$$

$[n]\varphi$:

"φ is provable in T together with the set of all true Π_n sentences."

Review: Worms

Worms : Formulas of the form

$$\langle n_1 \rangle \langle n_2 \rangle \ldots \langle n_m \rangle \top.$$

$\mathbb{W}$: The set of all worms.

Recursively:

- $\top$ is a worm
- if $\mathfrak{w}, \mathfrak{v}$ are worms, $\mathfrak{w}\, a\, \mathfrak{v}$ is a worm
- if $\mathfrak{w}$ is a worm and $a \in \mathbb{N}$ then $a \uparrow \mathfrak{w}$ is a worm

Where

- $(\langle x_1 \rangle \ldots \langle x_n \rangle \top)\, a (\langle y_1 \rangle \ldots \langle y_m \rangle \top) = \langle x_1 \rangle \ldots \langle x_n \rangle \langle a \rangle \langle y_1 \rangle \ldots \langle y_m \rangle \top$
- $a \uparrow \langle x_1 \rangle \ldots \langle x_n \rangle \top = \langle a + x_1 \rangle \ldots \langle a + x_n \rangle \top$

Review: Equivalences on worms

Lemma

- *If $a > b$ and ϕ, ψ are formulas then*

$$\mathrm{GLP} \vdash \langle a \rangle (\phi \wedge \langle b \rangle \psi) \leftrightarrow (\langle a \rangle \phi \wedge \langle b \rangle \psi).$$

- *If $\mathfrak{w} \in \mathbb{W}_{a+1}$ then*

$$\mathrm{GLP} \vdash \mathfrak{w} a \mathfrak{v} \leftrightarrow \mathfrak{w} \wedge a \mathfrak{v}$$

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Lemma

- *If* $a > b$ *and* ϕ, ψ *are formulas then*

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- *If* $\mathfrak{w} \in \mathbb{W}_{a+1}$ *then*

$$\mathrm{GLP} \vdash \mathfrak{w} a \mathfrak{v} \leftrightarrow \mathfrak{w} \wedge a\mathfrak{v}$$

- *If* $\mathrm{GLP} \vdash \mathfrak{w} \to \mathfrak{v}$ *then* $\mathrm{GLP} \vdash a \uparrow \mathfrak{w} \to a \uparrow \mathfrak{v}$

Measuring worms

$$\|\langle n_1\rangle\langle n_2\rangle\ldots\langle n_m\rangle\top\| = m + \max_{i\leq m} n_i$$

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Measuring worms

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If $\mathfrak{w} \neq \top$, there are $h(\mathfrak{w})$, $b(\mathfrak{w})$ such that

- $\mathfrak{w} \equiv (1 \uparrow h(\mathfrak{w}))0b(\mathfrak{w})$;
- $\|h(\mathfrak{w})\|, \|b(\mathfrak{w})\| < \|\mathfrak{w}\|$.

Ordering worms

Lemma
Worms are linearly preordered by

$$\mathfrak{v} <_0 \mathfrak{w} \Leftrightarrow \mathsf{GLP} \vdash \mathfrak{w} \rightarrow \Diamond \mathfrak{v}.$$

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$<_0$ recursively:

$\mathfrak{w} <_0 \mathfrak{v}$ whenever

- $\mathfrak{w} \leq_0 b(\mathfrak{v})$, or
- $b(\mathfrak{w}) <_0 \mathfrak{v}$ and $h(\mathfrak{w}) <_0 h(\mathfrak{v})$.

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Well-orders revisited

Definition

A pair $\langle A, \prec \rangle$ is a well-order if $\prec$ is a linear order on A satisfying any of the following equivalent conditions:

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- No bad sequence: There is no infinite sequence such that
$$a_0 \succ a_1 \succ a_2 \succ \dots .$$
- Minimal elements: Every non-empty $B \subseteq A$ has a $\prec$-minimum element.
- Transfinite induction: Let $\downarrow a = \{b \ : \ b \prec a\}$.
If $B \subseteq A$ has the property that, for all a, $\downarrow a \subseteq B \rightarrow a \in B$, then $B = A$.

Well-ordered worms

Theorem

The relation $<_0$ is a well-order on $\mathbb{W}$.

Proof that worms are well-ordered

$$\mathfrak{w}'_0 \quad >_0 \quad \mathfrak{w}'_1 \quad >_0 \quad \mathfrak{w}'_2 \quad >_0 \quad \mathfrak{w}'_3 \quad >_0 \quad \mathfrak{w}'_4$$

Towards a contradiction, assume there is an infinite descending chain.

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$$\mathfrak{w}_0 \quad >_0 \quad \mathfrak{w}'_1 \quad >_0 \quad \mathfrak{w}'_2 \quad >_0 \quad \mathfrak{w}'_3 \quad >_0 \quad \mathfrak{w}'_4$$

Fix $\mathfrak{w}_0$ so that $\|\mathfrak{w}_0\|$ is minimal.

Proof that worms are well-ordered

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Now, fix $\mathfrak{w}_1$ to minimize $\|\mathfrak{w}_1\|$.

Proof that worms are well-ordered

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Now, minimize $\|\mathfrak{w}_2\|$.

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And so on.

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Since $\|h(\mathfrak{w})\| < \|\mathfrak{w}\|$, $h(\mathfrak{w}_i) \leq_0 h(\mathfrak{w}_{i+1})$ for some i (say, $i = 2$)

Proof that worms are well-ordered

$$\mathfrak{w}_0 \quad >_0 \quad \mathfrak{w}_1 \quad >_0 \quad b(\mathfrak{w}_2) \quad ? \quad \mathfrak{w}_4 \quad >_0 \quad \mathfrak{w}_5$$

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$b(\mathfrak{w}_2) \leq_0 \mathfrak{w}_4 <_0 \mathfrak{w}_3$ by minimality of $\|\mathfrak{w}_2\|$.

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It follows that $\mathfrak{w}_2 \leq_0 \mathfrak{w}_3$!! $\square$

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It follows that $\mathfrak{w}_2 \leq_0 \mathfrak{w}_3$!! □

Corollary

There exists a function $o\colon \mathbb{W} \to \mathrm{Ord}$ *given by*

$$o(\mathfrak{w}) = \sup_{\mathfrak{v} <_0 \mathfrak{w}} o(\mathfrak{v})$$

Ordinal induction and recursion

Three types of ordinals:

- $\xi = 0$
- $\xi = \zeta + 1$
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Addition:

- $\xi + 0 = \xi$
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Multiplication:

- $\xi \cdot 0 = 0$
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- $\xi \cdot \lim_{\eta<\zeta} \eta = \bigcup_{\eta<\zeta} (\xi \cdot \eta)$

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Exponentiation:

- $\xi^0 = 1$
- $\xi^{\zeta+1} = \xi^\zeta \cdot \xi$
- $\xi^{\lim_{\eta<\zeta} \eta} = \lim_{\eta<\zeta} \xi^\eta$

The ordinal ε_0

Definition

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Theorem

The set ε_0 is an ordinal and satisfies the identity $\varepsilon_0 = \omega^{\varepsilon_0}$. Moreover, if $0 < \xi < \varepsilon_0$, there are $\alpha, \beta < \xi$ such that $\xi = \alpha + \omega^\beta$.

Computing orders below ε_0

Lemma

Given ordinals $\xi = \alpha + \omega^\beta$ and $\zeta = \gamma + \omega^\delta$,

1. $\xi < \zeta$ *if and only if*
   1.1 $\xi \leq \gamma$, *or*
   1.2 $\alpha < \zeta$ *and* $\beta < \delta$

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2. $\xi \leq \zeta$ *if and only if*
   2.1 $\xi \leq \gamma$, *or*
   2.2 $\alpha < \zeta$ *and* $\beta \leq \delta$.

Computing order-types of worms

Theorem

The function o is given recursively by

1. $o(\top) = 0$
2. $o\big((1 \uparrow \mathfrak{w})\, 0\, \mathfrak{v}\big) = o(\mathfrak{v}) + \omega^{o(\mathfrak{w})}$

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Proof sketch.

The map o as defined above is order-preserving and bijective, and there can be only one such map between well-orders. □

Arithmetical reflection principles

Statements of the form

"If φ is provable in T then φ is true."

Formally,

$$\Box_T \varphi \to \varphi.$$

- If φ is a sentence, this is an instance of local reflection.
- Uniform reflection generalizes this to formulas $\varphi = \varphi(x)$:

$$RFN_\varphi[T] = \forall x \big(\Box_T \varphi(\bar{x}) \to \varphi(x)\big).$$

Reflection schemes: $RFN_\Gamma[T] := \{RFN_\varphi[T] : \varphi \in \Gamma\}$.

Remark: By Löb's rule, T only proves its reflection instances when we already have that $T \vdash \varphi$.

Arithmetic through reflection

Theorem (Kreisel and Levy)

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Theorem (Leviant, Beklemishev)

For all $n \geq 1$, $I\Sigma_n \equiv \mathrm{EA} + \mathit{RFN}_{\Sigma_{n+1}}[\mathrm{EA}]$.

Reflection proves induction

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- If φ has unbounded quantifiers then EA cannot prove $I\varphi$ directly.

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  $\varphi(0) \land \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x)$.
- If φ has unbounded quantifiers then EA cannot prove $I\varphi$ directly.
- However, for any n, EA can prove that
  $\varphi(0) \land \forall x(\varphi(x) \to \varphi(x+1)) \to \varphi(\bar{n})$.

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- If φ has unbounded quantifiers then EA cannot prove $I\varphi$ directly.
- However, for any n, EA can prove that
  $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \varphi(\bar{n})$.
- EA can even prove this fact:
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Reflection proves induction

Reasoning in $\mathrm{EA} + RFN[\mathrm{EA}]$:

- Consider an instance $I\varphi$ of induction:
  $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x)$.
- If φ has unbounded quantifiers then EA cannot prove $I\varphi$ directly.
- However, for any n, EA can prove that
  $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \varphi(\bar{n})$.
- EA can even prove this fact:
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- By reflection we have $I\varphi$.

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All axioms of EA are true, and all rules preserve truth. Thus by induction on the length of a derivation, all theorems of EA are true.

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Solution: Cut elimination!

The Tait calculus

Sequent-based calculus, where all negations are pushed down to atomic formulas.

(LEM) $$\frac{}{\Gamma, \alpha, \neg\alpha}$$

($\wedge$) $$\frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}$$

($\vee$) $$\frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi}$$

($\forall$) $$\frac{\Gamma, \varphi(v)}{\Gamma, \forall x \varphi(x)}$$

($\exists$) $$\frac{\Gamma, \varphi(t)}{\Gamma, \exists x \varphi(x)}$$

(CUT) $$\frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma},$$

where α is atomic and v does not appear free in Γ.

Cut elimination

Theorem

It is provable in PA *that any sequent derivable in the Tait calculus can be derived without the cut rule.*

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In fact, we do not need full PA.

Let EA^+ be the theory $\mathrm{EA}+$"the superexponential is total".

Then, EA^+ suffices to prove cut-elimination.

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  where n is large enough so that all negated axioms of EA and φ are Π_n.
- Since all axioms of EA are provable in PA, we conclude that φ.

Reflection and *n*-consistency

We may also consider principles of the form $[n]_T\varphi \to \varphi$, or simply $\langle n\rangle_T\varphi := \neg[n]_T\neg\varphi$:

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Remark: The proof-theoretic ordinal of PA is

$$\sup_{n<\omega} o(\langle n\rangle\top) = \varepsilon_0$$

Topological semantics:

- GL-spaces: scattered topological spaces $\langle X, \mathcal{T} \rangle$
  Scattered: Every non-empty subset contains an isolated point.
- Valuations: dA is the set of limit points of A.

$$[\![\Diamond\varphi]\!] = d\,[\![\varphi]\!]\,.$$

GL is also sound and complete for both interpretations.

Some scattered spaces

- A finite partial order $\langle W, \geq \rangle$ with the downset topology
- An ordinal ξ with the initial segment topology
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Non-scattered:

- The real line
- The rational numbers
- The Cantor set

Review: Kripke semantics for GLP

Frames:

$$\langle W, \langle >_n \rangle_{n<\omega} \rangle$$

$[n]([n]\varphi \to \varphi) \to [n]\varphi$:

Valid iff $<_n$ is well-founded

$[n]\varphi \to [n+1]\varphi$:

Valid iff $w <_{n+1} v \Rightarrow w <_n v$

$\langle n \rangle\varphi \to [n+1]\langle n \rangle\varphi$:

Valid iff

$$v <_n w \text{ and } u <_{n+1} w \Rightarrow v <_n u$$

Even GLP_2 has no non-trivial Kripke models.

Topological semantics

Spaces:

$$\langle X, \langle \mathcal{T}_n \rangle_{n<\omega} \rangle$$

Write d_n for the limit point operator on $\mathcal{T}_n$.

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$[n]\varphi \to [n+1]\varphi$: Valid iff $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$

$\langle n \rangle \varphi \to [n+1]\langle n \rangle \varphi$: Valid iff

$$A \subseteq X \Rightarrow d_n A \in \mathcal{T}_{n+1}$$

Topological completeness

Beklemishev, Gabelaia: GLP is complete for the class of GLP-spaces

The proof uses non-constructive methods.

Blass: It is consistent with ZFC that the canonical ordinal spaces for GLP_2 are all trivial

Beklemishev: It is also consistent with ZFC that GLP_2 is complete for its canonical ordinal spaces

Bagaria More generally, for all n it is consistent with ZFC that GLP_n has non-trivial canonical ordinal spaces but GLP_{n+1} does not.

The closed fragment

Recall that the closed fragment is written GLP^0 and does not allow propositional variables (only $\perp$).

Beklemishev: GLP^0_ω may be used to perform ordinal analysis of PA, its natural subtheories and some extensions.

Theorem (Ignatiev)

There is a Kripke frame $\mathfrak{I}$ such that GLP^0_ω is sound and complete for $\mathfrak{I}$.

Ignatiev's model of GLP^0

Given an ordinal $\xi = \alpha + \omega^\beta$, define $\ell\xi = \beta$ ($\ell 0 = 0$).

Ignatiev's model:

$$\mathfrak{I} = \langle D, \langle >_n \rangle_{n<\omega} \rangle$$

- $D = \{f : \omega \to \varepsilon_0 : \forall n\, f(n+1) \leq \ell f(n)\}$
- $f <_n g$ if $f(m) = g(m)$ for $m < n$ and $f(n) < g(n)$

Example:

$$\langle \omega^{\omega+1}, \omega, 0, \ldots \rangle <_2 \langle \omega^{\omega+1}, \omega, 1, 0, \ldots \rangle$$

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$\langle n \rangle\varphi \to [n+1]\langle n \rangle\varphi$:

$$\begin{array}{ccc} \langle \omega^\omega, 0, 0, 0, \ldots \rangle & <_1 & \langle \omega^\omega, \omega, 1, 0, \ldots \rangle \\ \langle \omega^\omega, \omega, 0, 0, \ldots \rangle & <_2 & \langle \omega^\omega, \omega, 1, 0, \ldots \rangle \\ \hline \langle \omega^\omega, 0, 0, 0, \ldots \rangle & <_1 & \langle \omega^\omega, \omega, 0, 0, \ldots \rangle \end{array}$$

The main axis

Definition

A sequence $f : \omega \to \varepsilon_0$ is exact if for all n,

$$f(n+1) = \ell f(n).$$

Main axis: Set of exact sequences.

Lemma

Every closed formula which is satisfied on $\mathfrak{I}$ is satisfied on the main axis.

Is GLP^0 sound for $\mathfrak{I}$?

Icard topologies

Icard defined a structure

$$\mathfrak{I} = \langle \varepsilon_0, \langle \mathcal{T}_n \rangle_{n<\omega} \rangle.$$

Generlized intervals:

$$(\alpha, \beta)_n = \{\vartheta : \alpha < \ell^n \vartheta < \beta\}.$$

$\mathcal{T}_n$ is generated by intervals of the form

- $(\alpha, \beta)_m$ for $m < n$
- $[0, \beta)_m$ for $m \leq n$

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$\langle n\rangle\varphi \to [n+1]\langle n\rangle\varphi$: The point

$$\omega^\omega = \lim_{n\to\omega} \omega^n$$

should be isolated in $\mathcal{T}_2$.

Ignatiev vs. Icard

Define $\vec{\ell} : \varepsilon_0 \to D$ by

$$\vec{\ell}\xi = \langle \xi, \ell\xi, \ell^2\xi, \ldots, \ell^n\xi, \ldots \rangle$$

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Lemma

For every $\xi < \varepsilon_0$,

$$\langle \mathfrak{T}, \xi \rangle \models \varphi \Leftrightarrow \langle \mathfrak{I}, \vec{\ell}\xi \rangle \models \varphi$$

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For every $\xi < \varepsilon_0$,

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Corollary

$\mathfrak{I}$ and $\mathfrak{T}$ satisfy the same set of formulas.

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GLP^0 *is sound for both* $\mathfrak{I}$ *and* $\mathfrak{T}$.

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Theorem (Ignatiev, Icard)
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- Here we have Kripke models, simple topological models.
- Work in progress: Use modalities beyond ω to extend applications to stronger theories

FIN

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Thank you!