Provability Logic
Day 1
Modal logics of provability

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What is provability logic?

Idea: Given a formal theory T over a language L, we interpret $\Box\phi$ as

“ϕ is provable in T”.

In symbols we write $T \vdash \phi$.

This interpretation of modal logic was first suggested by Kurt Gödel.

It can be used to reason about Gödel’s famous incompleteness theorems.

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- $\vdash \phi \leftrightarrow \neg\Box\phi$: Liar paradox
- $\Diamond\top \rightarrow \Diamond\Box\bot$: Second incompleteness theorem

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Henkin asks, “What can we say about formulas that assert their own provability”?

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Löb's rule: $\dfrac{\Box\phi \rightarrow \phi}{\phi}$

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$$\forall\psi\exists\phi \mathsf{GL} \vdash \phi \leftrightarrow \psi(\phi)$$

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$$\text{GL} \vdash \phi \Leftrightarrow \forall f\big(\mathbb{N} \models f(\phi)\big)$$

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No Kripke models!

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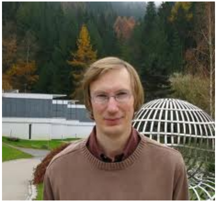
In 2004, Lev Beklemishev showed how Japaridze's system GLP_ω can be used to give an ordinal analysis of Peano Arithmetic.



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- 2012 Beklemishev and Dashkov propose the less expressive but more efficient Reflection Calculus
- 2012-2017 DFD and Joosten extend the above results to systems with transfinite modalities

Basic ingredients

We will need:

1. A formal language L to speak about arithmetic.
2. A formal theory T that reasons about arithmetic
3. A provability predicate $\mathtt{Prv}_T$ which talks about provability within L
4. A modal logic where $\Box \approx \mathtt{Prv}_T$

Arithmetical languages

An arithmetical interpretation of a first- or higher-order language L is an L-model $\mathfrak{N} = \langle \mathbb{N}, I \rangle$ such that:

- there is an L-term $\texttt{0}$ such that $I(\texttt{0}) = 0$
- there is a unary function symbol S such that for all $n \in \mathbb{N}$, $I(\dot{n}) = n$, where
$$\dot{n} = \underbrace{\texttt{SS}\ldots\texttt{S}}_{n}\,\texttt{0}$$
- there are binary function symbols $\texttt{plus}$, $\texttt{times}$, $\texttt{exp}$ such that, given $n, m \in \mathbb{N}$,
  - $I\big(\texttt{sum}(\dot{n}, \dot{m})\big) = n + m$
  - $I\big(\texttt{times}(\dot{n}, \dot{m})\big) = n \times m$
  - $I\big(\texttt{exp}(\dot{n}, \dot{m})\big) = n^m$

  We will usually write $\mathbb{N} \models \phi$ instead of $\mathfrak{N} \models \phi$.

The arithmetical hierarchy

A bounded quantifier is one that appears in a context $\forall x(x < t \rightarrow \phi)$ or $\exists x(x < t \wedge \phi)$.

A formula ϕ is elementary or Δ_0 if all quantifiers appearing in ϕ are bounded.

Then, define by induction:

- $\Pi_0 = \Sigma_0 = \Delta_0$
- if $\phi \in \Sigma_n$ then $\forall x_0 \forall x_1 \ldots \forall x_m \phi \in \Pi_{n+1}$
- if $\phi \in \Pi_n$ then $\exists x_0 \exists x_1 \ldots \exists x_m \phi \in \Sigma_{n+1}$

Fact: Every first-order formula is provably equivalent in FOL to either a Π_n-formula or a Σ_n-formula.

Gödel numbers

A Gödel numbering is an assignment $\phi \mapsto \ulcorner\phi\urcorner$ mapping an L-formula to a natural number.

This allows us to reason about formal languages within arithmetic.

There are many ways to do this:

- ASCII code
- products of prime powers
- using the Chinese remainder theorem

Substitution

Many standard syntactic operations are elementary and hence can be represented by a Δ_0 formula.

Proposition

In any arithmetical language L *there is a* Δ_0 *formula* $\mathtt{subs}(\mathtt{w}, \mathtt{x}, \mathtt{y}, \mathtt{z})$ *such that for all tuples of natural numbers* a, b, n, m*,*

$$\mathbb{N} \models \mathtt{subs}(\dot{a}, \dot{b}, \dot{n}, \dot{m})$$

if and only if there is is a formula α*, a term* t *and a variable* v *with*

$$a = \ulcorner \alpha \urcorner \qquad n = \ulcorner t \urcorner \qquad m = \ulcorner v \urcorner$$

and

$$b = \ulcorner \alpha[x/t] \urcorner.$$

Formalized substitution is crucial to Gödel's proof and Löb's fixpoint theorem which we will see later.

Formal theories

A formal theory *T* is usually presented as a family of *rules* and *axioms*.

Definition

A derivation of ϕ is a sequence $\langle \phi_0, \ldots \phi_N \rangle$ such that $\phi_N = \phi$ and each ϕ_n is either an axiom or follows by the rules from $\phi_0, \ldots, \phi_{n-1}$.

If ϕ is derivable in *T* we write $T \vdash \phi$.

All theories will be assumed closed under generalization and modus ponens:

$$\frac{\phi}{\forall x \phi} \qquad \frac{\phi \qquad \phi \to \psi}{\psi}$$

Arithmetical theories

L is an arithmetically interpreted language, T is a theory over L.

Definition

The theory T is arithmetically sound if whenever T $\vdash \phi$, $\mathbb{N} \models \phi$
The theory T is arithmetically complete if, whenever $\mathbb{N} \models \phi$, T $\vdash \phi$.

There are also relative versions of these notions. For example, if Γ is a set of formulas, *T* is Γ-sound if every theorem of *T* that also belongs to Γ is true.

We will be mainly interested in arithmetically sound theories.

Robinson arithmetic Q

Axiomatized by FOL and:

- $\forall \mathtt{x}(\mathtt{x} = \mathtt{x})$
- $\forall \mathtt{x} \forall \mathtt{y} \forall \mathtt{z}(\mathtt{x} = \mathtt{y} \wedge \mathtt{y} = \mathtt{z} \to \mathtt{x} = \mathtt{z})$
- $\forall \mathtt{x} \forall \mathtt{y}(\mathtt{x} = \mathtt{y} \leftrightarrow \mathtt{S}\mathtt{x} = \mathtt{S}\mathtt{y})$
- $\neg \exists \mathtt{x}(\mathtt{0} = \mathtt{S}\mathtt{x})$
- $\forall \mathtt{x}\, \mathtt{sum}(\mathtt{x}, \mathtt{0}) = \mathtt{x}$
- $\forall \mathtt{x} \forall \mathtt{y}(\mathtt{sum}(\mathtt{x}, \mathtt{S}\mathtt{y}) = \mathtt{S}(\mathtt{sum}(\mathtt{x} + \mathtt{y})))$
- $\forall \mathtt{x}\, \mathtt{times}(\mathtt{x}, \mathtt{0}) = \mathtt{0}$
- $\forall \mathtt{x} \forall \mathtt{y}\, \mathtt{times}(\mathtt{x}, \mathtt{S}\mathtt{y}) = \mathtt{sum}(\mathtt{times}(\mathtt{x}, \mathtt{y}), \mathtt{x})$
- $\forall \mathtt{x}\, \mathtt{exp}(\mathtt{x}, \mathtt{0}) = \mathtt{S}\mathtt{0}$
- $\forall \mathtt{x} \forall \mathtt{y}\, \mathtt{exp}(\mathtt{x}, \mathtt{S}\mathtt{y}) = \mathtt{times}(\mathtt{exp}(\mathtt{x}, \mathtt{y}), \mathtt{x})$

Theories with induction

Induction (Ind): $\phi(0) \wedge \forall x(\phi \to \phi(Sx)) \to \forall x \phi(x)$.

Peano arithmetic (PA): $\mathsf{Q} + \mathrm{Ind}$

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Peano arithmetic (EA): $Q + \mathrm{Ind}(\Delta_0)$

The fixpoint lemma

Proposition

Given an arithmetical formula $\psi(\mathtt{n}, \tilde{\mathtt{x}})$ *an arithmetical formula, there exists a formula* $\mathtt{G}$ *such that*

$$\mathsf{EA} \vdash \forall \mathtt{x} \mathtt{G} \leftrightarrow \psi(\ulcorner \mathtt{G} \urcorner, \tilde{\mathtt{x}})$$

Elementarily presented theories

Reasonable requirement: The axioms and rules are recursively enumerable.

Syntactically, they can be described by a Σ_1-formula.

If all axioms and rules of T are Δ_0 we say that T is elementarily presented.

Craig's trick: If the axioms and rules of T are Σ_1-definable, then there is an elementarily presented family of axioms and rules which give the same theorems as T.

Provability predicates

Derivations can be assigned Gödel numbers just like formulas, which allows us to study logic within any arithmetically interpreted language.

Proposition (Gödel)

If T is elementarily presented there is a Δ_0-formula $\mathtt{prv}(\mathtt{x}, \mathtt{y})$ such that for all $n, m \in \mathbb{N}$, $\mathbb{N} \models \mathtt{prv}_T(\dot{n}, \dot{m})$ if and only if there is a derivation d of a formula ϕ with $n = \ulcorner \phi \urcorner$ and $m = \ulcorner d \urcorner$.

With this we can define

- ϕ is provable in T:

$$\mathtt{Prv}(\mathtt{x}) := \exists \mathtt{y}\, \mathtt{prv}(\mathtt{x}, \mathtt{y}).$$

- T is consistent:

$$\mathtt{Cons}(T) := \neg \mathtt{Prv}(\ulcorner \bot \urcorner)$$

Gödel's theorems

Theorem (First incompleteness theorem)

No elementarily presentable theory is arithmetically sound and complete.

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Theorem (Second incompleteness theorem)

If an elementarily presentable theory T extending EA *is arithmetically sound, then*

$$T \nvdash \mathtt{Cons}(T).$$

Tarskian truth predicates

Theorem

There is no formula $\texttt{True}(x)$ *in the language of* PA *such that for all* ϕ*,*

$$\mathbb{N} \models \phi \leftrightarrow \texttt{True}(\ulcorner \phi \urcorner)$$

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Proof.

Apply the fixed point theorem to obtain

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For every n *there is a formula* $\mathtt{True}_{\Pi_n}(x)$ *such that for all* $\phi(z) \in \Pi_n$,

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Note: We can define $\mathtt{True}_{\Sigma_n}(x) = \neg \mathtt{True}_{\Pi_n}(\neg x)$

Arithmetical realizations

Definition

An arithmetical interpretation is a function $f : \mathsf{L}_{\mathsf{GL}} \to \mathsf{L}_{\mathsf{PA}}$ such that:

- p^f is any sentence;
- f commutes with Booleans;
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Note: PA can be replaced by a stronger theory such as ZFC, or a weaker theory, such as Elementary arithmetic (EA).

Relational semantics

Definition

A (strictly) partially ordered set is a pair $\langle A, \succ \rangle$ where $\succ$ is a transitive, irreflexive relation on A.

The relation $\succ$ is well-founded if there is no infinite sequence such that

$$a_0 \succ a_1 \succ a_2 \succ \ldots$$

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GL-frame: $\langle W, \succ \rangle$ where $\succ$ is a well-founded partial order on W.

Relational models

GL-model: GL-frame $\langle W, \succ, [\![\cdot]\!] \rangle$ equipped with a valuation $[\![\cdot]\!] : \mathsf{L}_{\mathsf{GL}} \to 2^W$ such that

- $[\![p]\!] \subseteq W$ arbitrary
- $[\![\neg\varphi]\!] = W \setminus [\![\varphi]\!]$
- $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$
- $w \in [\![\Box\varphi]\!] \Leftrightarrow \forall v \prec w \, (v \in [\![\varphi]\!])$

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Theorem (Segerberg)

The following are equivalent:

- $\mathsf{GL} \vdash \varphi$
- *for every (finite)* GL*-model* $\langle W, \prec, [\![\cdot]\!] \rangle$, $[\![\varphi]\!] = W$.

Arithmetical completeness

GL is complete for its arithmetical interpretation.

Theorem (Solovay, 1976)

If ϕ is a GL formula such that $\mathrm{PA} \vdash f(\phi)$ for every arithmetical realization f, then $\mathrm{GL} \vdash \phi$.

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If ϕ is a GL *formula such that* PA $\vdash f(\phi)$ *for every arithmetical realization f, then* GL $\vdash \phi$*.*

Proof sketch.

1. Assume ϕ is a consistent GL formula.
2. Pick a GL-model $\mathfrak{M}$ satisfying ϕ.
3. For each world w of $\mathfrak{M}$ build a formula $\theta(w)$ 'emulating' $\mathfrak{M}$.
4. Define $f(p) = \bigvee_{w \in V(p)} \theta(w)$.

□

Chasing the jewell thief

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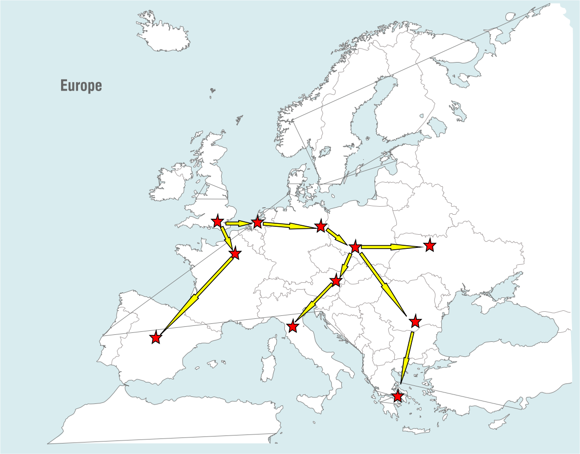
- she will never return to where she's already been
- her next stop will be the most unexpected
- in fact, she will only move to a new country if we can prove she will not go there!

The train routes



The thief in a Kripke frame

We have a GL-model $\mathfrak{M} = \langle W, \succ, V \rangle$ such that

- $W = \{1, 2, \ldots, m\}$
- $\succ$ is a partial order
- 1 is the root (maximum element)

We add an "imaginary root" 0.

Chasing the thief

On day 0, we begin at the world 0, and will follow a path

$$w_0, w_1, \ldots w_n.$$

The formula $\theta(v)$ asserts "$w_n = v$"

The world v is the most unexpected possible world on day $i + 1$ if

1. $w_i \succ v$
2. i codes a proof of $\neg\theta(v)$.

The thief will stop her journey at w_n, from which there are no unexpected worlds.

The Solovay function

$$\psi(k, n, w) =$$

$$\bigvee_{v \succ w} \mathtt{True}_{\Sigma_1}(k(n, w)) \wedge \mathtt{dem}_{\mathsf{PA}}(n, \ulcorner \neg \forall x \exists y \dot{>} x. k(y, w) \urcorner)$$

Fixpoint theorem: there is a formula $H(n, w)$ such that for all n, w

$$H(n, w) \leftrightarrow \psi(\ulcorner H \urcorner, n, w)$$

We define $h(n)$ as the unique w such that $\mathbb{N} \models H(n, w)$

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- if $w \neq 0$, $\mathrm{PA} + Lim(w)$ proves $\Box_{\mathrm{PA}} \bigvee_{v \prec w} Lim(v)$

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- $\mathbb{N} \models h(n) \equiv 0$

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Proposition

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- $\mathfrak{M}, 0 \models \Diamond\psi \Rightarrow \mathrm{PA} + Lim(0) \vdash f(\Diamond\psi)$

Since $\mathbb{N} \models \mathrm{PA} + Lim(0)$, we conclude that $f(\varphi)$ is consistent.

Polymodal Gödel-Löb (GLP)

Japaridze, 1988: Extend GL to have one modality $[n]$ for each $n < \omega$.

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Axioms: all tautologies, plus

$$[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi) \qquad (n < \omega)$$

$$[n]([n]\varphi \to \varphi) \to [n]\varphi \qquad (n < \omega)$$

Polymodal Gödel-Löb (GLP)

Japaridze, 1988: Extend GL to have one modality $[n]$ for each $n < \omega$.

Axioms: all tautologies, plus

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Rules: Modus ponens and necessitation for all modalities

Arithmetical interpretations

Recall: Π_n formulas are arithmetical formulas of the form

$$\forall x_1 \exists x_2 \forall x_3 \ldots Q_n x_n \varphi$$

If $f : \mathrm{L}_{\mathrm{GL}} \to \mathrm{L}_{\mathrm{PA}}$, we extend f to all $\mathrm{L}_{\mathrm{GLP}}$:

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- for $n > 0$, $f([n]\varphi)$ means
  "Provable in T together with the set of all true Π_n sentences."

Interpretation of $[n]$

Recall: For all $n < \omega$ there is a formula $\mathtt{True}_n(x)$ such that $\mathbb{N} \models \mathtt{True}_n(\overline{k})$ if and only if k codes a true Π_n-sentence.

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Theorem (Japaridze, Ignatiev)

The following are equivalent:

- GLP $\vdash \varphi$
- *for every arithmetical interpretation f,* PA $\vdash \varphi^f$.

Relational semantics

Frames:

$$\langle W, \langle \succ_n \rangle_{n<\omega} \rangle$$

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Even GLP_2 has no non-trivial Kripke models.

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Beklemishev: RC^0 may be used to perform ordinal analysis of PA, its natural subtheories and some extensions.

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Wrm : the class of all worms

Wrm_n : the class of all worms with entries at least n

Basic equivalences

Lemma

- *If $a > b$ and ϕ, ψ are formulas then*

$$\mathrm{GLP} \vdash \langle a \rangle (\phi \wedge \langle b \rangle \psi) \leftrightarrow (\langle a \rangle \phi \wedge \langle b \rangle \psi).$$

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- *If* $\mathrm{w} \in S_{a+1}$ *then*

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Worms recursively

- $\top$ is a worm
- if w, v are worms, w 0 v is a worm
- if w is a worm and $a \in \mathbb{N}$ then $a \uparrow$ w is a worm

Where

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Tomorrow: w 0 v $\approx$ w + v, $1 \uparrow$ v $\approx \omega^{\mathrm{v}}$

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