Forcing interpretation, conservation and proof size

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In this talk,

- mainly consider relational languages without equality,
- use a fixed proof system (LK, NK or any other reasonable system).

* We are not restricted in our context even if we only use relational languages.

The major part of this talk is an overview of Avigad [1,2].

1. J. Avigad, Forcing in proof theory. Bull. Symbolic Logic 10 (2004), no. 3, 305-333.
2. J. Avigad, Formalizing forcing arguments in subsystems of second-order arithmetic. Ann. Pure Appl. Logic 82 (1996), no. 2, 165-191.
3. L. A. Kolodziejczyk, T. L. Wong and K. Yokoyama, Ramsey's theorem for pairs, collection, and proof size, submitted.

The precise definition of forcing interpretation (with function symbols) is reorganized in [3].

Relative translation

Definition

A (one-dimentional) relative translation τ from a language $\mathcal{L}'$ to another language $\mathcal{L}$ consists of the following:

- $\mathcal{L}$-formula $\varphi_D(x)$: domain of an $\mathcal{L}'$-structure
- $\mathcal{L}$-formula $\varphi_R(\vec{x})$ for each $R \in \mathcal{L}'$: interpretation of R

If $\mathcal{M} = (M; \dots)$ is an $\mathcal{L}$-structure, then $(D^\tau; R^\tau, \dots)$ is an $\mathcal{L}'$-structure where $D^\tau = \{a \in M : \mathcal{M} \models \varphi_D(a)\}$, $R^\tau = \{\vec{a} \in D^\tau : \mathcal{M} \models \varphi_R(\vec{x})\}, \dots$

In this sense, any $\mathcal{L}'$-formula ψ can be translated to an $\mathcal{L}$-formula ψ^τ by relativization, i.e., by formalizing Tarski's truth definition for $(D^\tau; R^\tau, \dots) \models \psi$.

Relative interpretation

Let T be an $\mathcal{L}$-theory and T' be an $\mathcal{L}'$-theory.

Definition

A relative translation τ from $\mathcal{L}'$ to $\mathcal{L}$ is said to be a relative interpretation of T' in T if $T \vdash \psi^\tau$ for each $\psi \in T'$.

By formalizing the usual soundness proof (by induction on the complexity of formulas), we have the following.

Theorem (Soundness theorem)

If τ is a relative interpretation of T' in T and $T' \vdash \theta$, then $T \vdash \theta^\tau$.

Corollary

If τ is a relative interpretation of T' in T, then $\mathrm{Con}(T)$ implies $\mathrm{Con}(T')$.

Relative interpretation behaves well in the following sense.

Let T be an $\mathcal{L}$-theory, T' be an $\mathcal{L}'$-theory and Γ be a class of $\mathcal{L} \cap \mathcal{L}'$-formulas.

Definition

A relative interpretation τ is said to be Γ-reflecting if

$$T \vdash \psi^\tau \to \psi \text{ for any } \psi \in \Gamma.$$

Theorem

If there exists a relative interpretation of T' in T which is Γ-reflecting, then T' is Γ-conservative over T.

Polynomial reflection and proof size

Let T be an $\mathcal{L}$-theory, T' be an $\mathcal{L}'$-theory and Γ be a class of $\mathcal{L} \cap \mathcal{L}'$-formulas.

Definition

A relative translation τ is said to be a polynomial interpretation of T' in T if there is polynomial-time procedure which, given any $\psi \in T'$, outputs a proof of $T \vdash \psi^\tau$.

A relative interpretation τ is said to be polynomially Γ-reflecting if there is polynomial-time procedure which, given any $\theta \in \Gamma$, outputs a proof of $T \vdash \theta^\tau \to \theta$.

Note that any relative interpretation of a finite theory T is a polynomial interpretation.

Theorem

If there exists a polynomial relative interpretation of T' in T which is polynomially Γ-reflecting, then T polynomially simulates T' with respect to Γ, i.e., there is polynomial-time procedure which, given any proof of $T' \vdash \theta$ for $\theta \in \Gamma$, outputs a proof of $T \vdash \theta$.

Example: RCA_0 vs $\mathrm{I}\Sigma_1$

Let $n \geq 1$. A relative translation τ_{REC} from $\mathcal{L}_2$ to $\mathcal{L}_1$ consists of the following:

- $\varphi_M(x) :\equiv x = x$,
- $\varphi_S(e) :\equiv$ "e is an index of a Δ_1-set",
- $\varphi_\in(x, e) :\equiv$ "x is an element of the e-th Δ_1-set".

Proposition

- τ_{REC} *is an interpretation of* $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n$ *in* $\mathrm{I}\Sigma_n$.
- τ_{REC} *is polynomially* $\mathcal{L}_1$*-reflecting in* $\mathrm{I}\Sigma_1$.

Corollary

$\mathrm{I}\Sigma_n$ *polynomially simulates* $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n$ *w.r.t.* $\mathcal{L}_1$*-sentences.*

Similarly, $\mathrm{I}\Sigma^0_n$ polynomially simulates $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_n$ w.r.t. Π^1_1-sentences.

Kripke semantics

A Kripke model is a quadraple $\mathcal{K} = (K, \leq_K, D, \Vdash^+)$, where

- $(K, \leq_K)$ is a pre-ordered set,
- $D = \{D_k\}_{k \in K}$ is a family of nonempty sets such that $D_k \subseteq D_{k'}$ if $k \leq_K k'$,
- $\Vdash^+$ is a relation, called a valuation, from K to the set of atomic formulae of the language extended by adding a constant symbols a for each element $a \in \bigcup\{D_k \mid k \in K\}$ such that
  - $k \Vdash^+ R(a_1, \dots, a_n) \Rightarrow a_i \in D_k$ for $i \in \{1, \dots, n\}$,
  - $k \Vdash^+ R(a_1, \dots, a_n)$ and $k \leq k' \Rightarrow k' \Vdash^+ R(a_1, \dots, a_n)$

  for each $k, k' \in K$.
- The valuation $\Vdash^+$ is then extended to any formulas by the following clauses
  1. $k \nVdash^+ \bot$,
  2. $k \Vdash^+ \varphi \wedge \psi \Leftrightarrow k \Vdash^+ \varphi$ and $k \Vdash^+ \psi$,
  3. $k \Vdash^+ \varphi \vee \psi \Leftrightarrow k \Vdash^+ \varphi$ or $k \Vdash^+ \psi$,
  4. $k \Vdash^+ \varphi \to \psi \Leftrightarrow k' \Vdash^+ \varphi$ implies $k' \Vdash^+ \psi$ for each $k' \geq k$.
  5. $k \Vdash^+ \forall x \varphi \Leftrightarrow \forall k' \geq k \forall a \in D_{k'} (k' \Vdash^+ \varphi[x/a])$,
  6. $k \Vdash^+ \exists x \varphi \Leftrightarrow \exists a \in D_k (k \Vdash^+ \varphi[x/a])$.

It is well-known that the Kripke semantics is sound and complete for intuitionistic predicate calculus.
Here, we focus on classical logic.

Define a new relation $\Vdash$ by

$$k \Vdash \psi \iff k \Vdash^{+} \neg\neg\psi.$$

Then, we have

Proposition (Soundness and completeness)

The following are equivalent.

1. $T \vdash \psi$ *(in classical logic).*
2. $\mathcal{K} \Vdash \psi$ *implies* $\mathcal{K} \Vdash \psi$ *for any Kripke model* $\mathcal{K}$.

We consider interpretation with this semantics.

Translation by Kripke semantics/ forcing translation

Definition

A forcing translation τ from $\mathcal{L}'$ to $\mathcal{L}$ consists of the following:

- $\mathcal{L}$-formula $\varphi_K(k)$: domain of a pre-order,
- $\mathcal{L}$-formula $\varphi_\leq(k, k')$: pre-order on K,
- $\mathcal{L}$-formula $\varphi_D(x, k)$: $\varphi_D(\cdot, k)$ defines the domain at k,
- $\mathcal{L}$-formula $\varphi_R(\vec{x}, k)$ for each $R \in \mathcal{L}'$: valuation of R at k.

If $\mathcal{M} = (M; \dots)$ is an $\mathcal{L}$-structure, then $\mathcal{K}^\tau = (K^\tau, \leq^\tau, D_k^\tau, \Vdash^+)$ is a Kripke model for $\mathcal{L}'$ if

- $D_k^\tau = \{a \in M : \mathcal{M} \models \varphi_D(a, k)\}$,
- $k \Vdash^+ R(\vec{a}) \Leftrightarrow \vec{a} \in D_k \wedge M \models \varphi_R(\vec{a}, k)$,
- $\varphi_\leq(k, k')$ defines a pre-order,
- $D_k \subseteq D_{k'}$ if $\varphi_\leq(k, k')$.

$a \in \bigcup_{k \in K^\tau}$ is often called a name, and we write $k \Vdash a \downarrow$ if $\varphi_D(a, k)$.

- For any $\mathcal{L}'$-formula ψ, the truth "$k \Vdash \theta$" can be described by an $\mathcal{L}$-formula, which gives a translation.

Interpretation by Kripke semantics/ forcing interpretation

Let T be an $\mathcal{L}$-theory and T' be an $\mathcal{L}'$-theory.

Definition

A forcing translation τ from $\mathcal{L}'$ to $\mathcal{L}$ is said to be a forcing interpretation of T' in T if T proves

- $\varphi_{\leq}(k, k')$ defines a pre-order,
- $D_k \subseteq D_{k'}$ if $\varphi_{\leq}(k, k')$,
- $k \Vdash a \downarrow$ if and only if $\forall k' \geq k \exists k'' \geq k' (k'' \Vdash a \downarrow)$,

and

- $T \vdash\Vdash \psi$ for each $\psi \in T'$.
  (Here, $\Vdash \theta$ means that $\forall k(\varphi_K(k) \to k \Vdash \theta)$).

By formalizing the soundness proof, we have the following.

Theorem (Soundness theorem)

If τ is a forcing interpretation of T' in T and $T' \vdash \theta$, then $T \vdash\Vdash \theta$.

Reflection and conservation

Forcing interpretation is well-behaved as relative interpretation.

Let T be an $\mathcal{L}$-theory, T' be an $\mathcal{L}'$-theory and Γ be a class of $\mathcal{L} \cap \mathcal{L}'$-formulas.

Definition

A forcing interpretation τ is said to be Γ-reflecting if

$$T \vdash\Vdash^{\tau} \psi \to \psi \text{ for any } \psi \in \Gamma.$$

Theorem

If there exists a forcing interpretation of T' in T which is Γ-reflecting, then T' is Γ-conservative over T.

Polynomial reflection and proof size

Let T be an $\mathcal{L}$-theory, T' be an $\mathcal{L}'$-theory and Γ be a class of $\mathcal{L} \cap \mathcal{L}'$-formulas.

Definition

A forcing translation τ is said to be a polynomial interpretation of T' in T if there is polynomial-time procedure which, given any $\psi \in T'$, outputs a proof of $T \vdash\Vdash^{\tau} \psi$.

A forcing interpretation τ is said to be polynomially Γ-reflecting if there is polynomial-time procedure which, given any $\theta \in \Gamma$, outputs a proof of $T \vdash\Vdash^{\tau} \theta \to \theta$.

Note that any forcing interpretation of a finite theory T is a polynomial interpretation.

Theorem

If there exists a polynomial forcing interpretation of T' in T which is polynomially Γ-reflecting, then T polynomially simulates T' with respect to Γ.

Questions

- $T' \leq_{\mathrm{rel}} T :\Leftrightarrow$ there exists a relative interpretation of T' in T
- $T' \leq_{\mathrm{f}} T :\Leftrightarrow$ there exists a forcing interpretation of T' in T

Question

Is $\leq_{\mathrm{f}}$ different from $\leq_{\mathrm{rel}}$?

Is forcing interpretation strong enough to cover all conservation/ non-speedup proofs?

Question

If a theory T' is Γ-conservative over a theory T, then does there always exist a forcing interpretation of T' in T which is Γ-reflecting?

Question

If a theory T polynomially simulates T' w.r.t. Γ-sentences, then does there always exist a polynomial forcing interpretation of T' in T which is polynomially Γ-reflecting?

Example 1: forcing with low sets

In the study of second-order arithmetic, Π^1_1-conservation theorems for Π^1_2-theories are often obtained by formalizing “low-basis theorems” in computability theory.

- low-basis theorem for $\mathrm{WKL} \Rightarrow \Pi^1_1$-conservation of WKL over $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2$
- low$_2$-basis theorem for $\mathrm{RT}^2 \Rightarrow \Pi^1_1$-conservation of RT^2 over $\mathsf{RCA}_0 + \mathrm{I}\Sigma^0_2$
- low-basis theorem for $\mathrm{SADS}, \mathrm{SCAC}, \dots$

It is known (or believed?) that

> if a low-basis theorem for a Π^1_2-statement $\forall X \exists Y \theta(X, Y)$ is provable within $\mathsf{RCA}_0 + \mathrm{I}\Sigma^0_n$ (or $\mathrm{B}\Sigma^0_n$) “EFFECTIVELY”, its iteration is also provable effectively, and . . . ,
> then the standard Π^1_1-conservation proof by constructing ω-extension can be reformulated with relative interpretation, and thus polynomial simulation w.r.t. Π^1_1-sentences is available(??)

In $\mathrm{I}\Sigma^0_1$, Turing reduction is formalizable, and thus Turing jump, low_n-sets, ... are available. Write $W^n[e, X]$ for the e-th Δ^X_n-set.

Let $n \geq 1$. A forcing translation $\tau(\mathrm{Low}_{n-1,X})$ from $\mathcal{L}_2$ to $\mathcal{L}_2$ consists of the following:

- the set of conditions $\mathbb{P} = \mathrm{Low}_{n-1,X}$ consists of all pairs of the form $\langle e, X\rangle$ such that e is an index of a low^X_{n-1}-set,
- $\langle e, X\rangle \geq_{\mathbb{P}} \langle f, Y\rangle$ if $X = Y$ and $W[e, X] \geq_T W[f, Y]$,
- names for numbers are numbers $v \in \mathbb{N}$,
- names for sets are conditions $\langle e, X\rangle \in \mathbb{P}$,
- $\langle e, X\rangle \Vdash v\downarrow$ always, and $\langle e, X\rangle \Vdash \langle f, Y\rangle\downarrow$ if $\langle f, Y\rangle \leq_{\mathbb{P}} \langle e, X\rangle$,
- $\langle e, X\rangle \Vdash v \in \langle f, Y\rangle$ if $\langle f, Y\rangle\downarrow$ and $v \in W[f, Y]$.

Proposition

$\tau(\mathrm{Low}_{n-1,X})$ is polynomially Π^1_1-reflecting over RCA_0.

Example 1: forcing with low sets

Theorem

Let $n \geq 1$. Let $\Theta \equiv \forall X \exists Y \theta(X, Y)$ be a Π^1_2-sentence.

1. *If* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_n$ *proves*
   $\forall X_0 \forall X \leq_T X_0 \exists e \in \mathsf{Low}_{n-1,X_0} \theta(X, W^n[e, X_0])$,
   then $\tau(\mathsf{Low}_{n-1,X})$ *is a forcing interpretation of* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_n + \Theta$ *in* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_n$.
2. *If* $\mathsf{RCA}^*_0 + \mathsf{B}\Sigma^0_n + \exp$ *proves*
   $\forall X_0 \forall X \leq_T X_0 \exists e \in \mathsf{Low}_{n-1,X_0} \theta(X, W^n[e, X_0])$,
   then $\tau(\mathsf{Low}_{n-1,X})$ *is a forcing interpretation of* $\mathsf{RCA}^*_0 + \mathsf{B}\Sigma^0_n + \Theta$ *in* $\mathsf{RCA}^*_0 + \mathsf{B}\Sigma^0_n$.

Corollary

- *If* $k \geq 2$, $\mathsf{I}\Sigma^0_k$ *polynomially simulates* $\mathrm{WKL}_0 + \mathsf{I}\Sigma^0_k$ *and* $\mathsf{B}\Sigma^0_k$ *polynomially simulates* $\mathrm{WKL}_0 + \mathsf{B}\Sigma^0_k$ *w.r.t.* Π^1_1.
- *If* $k \geq 3$, $\mathsf{I}\Sigma^0_k$ *polynomially simulates* $\mathrm{WKL}_0 + \mathrm{RT}^2 + \mathsf{I}\Sigma^0_k$ *and* $\mathsf{B}\Sigma^0_k$ *polynomially simulates* $\mathrm{WKL}_0 + \mathrm{RT}^2 + \mathsf{B}\Sigma^0_k$ *w.r.t.* Π^1_1.

Example 2: forcing for WKL revisited

Avigad used forcing interpretation to show that RCA_0 polynomially simulates WKL_0 with respect to Π^1_1-sentences.
Can we improve this?

Let $\Gamma_{\mathrm{STY}} = \{\forall X \exists ! Y \alpha(X, Y) : \alpha \text{ is arithmetical}\}$.

Theorem (Simpson/Tanaka/Yamazaki)

WKL_0 *is* Γ_{STY}*-conservative over* RCA_0.

Question

1. (Tanaka) Does RCA_0 polynomially simulate WKL_0 with respect to Γ_{STY}-sentences.
2. (Wong) Is WKL_0^* Γ_{STY}-conservative over RCA_0^*?

Proposition (RCA_0^*)

For any X, there exists a Δ^X_1-tree $\mathcal{T}^X$ such that any path $\mathcal{W} \in [\mathcal{T}^X]$ forms a countable coded ω-model of WKL with $\mathcal{W}_0 = X$.

A forcing translation τ consists of the following:

- the set of conditions $\mathbb{P}$ is the set of all pairs of the form $\langle X, T\rangle$ where T is (an index of) a Δ^X_1-definable infinite subtree of $\mathcal{T}^X$,
- for given $\langle X, T\rangle, \langle Y, U\rangle \in \mathbb{P}$, $\langle X, T\rangle \geq_{\mathbb{P}} \langle Y, U\rangle$ if $X = Y$ and $T \subseteq U$,
- names for numbers are numbers $v \in \mathbb{N}$,
- names for sets are numbers $V \in \mathbb{N}$,
- $\langle X, T\rangle \Vdash^\tau v\downarrow$, $\langle X, T\rangle \Vdash^\tau V\downarrow$ for any $\langle X, T\rangle \in \mathbb{P}$ and names v, V,
- $\langle X, T\rangle \Vdash^\tau v \in V$ if for any $\sigma \in T$, $v < |\sigma_V| \to \sigma_V(v) = 1$.

Example 2: forcing for WKL revisited

Proposition

1. $\mathsf{RCA}_0^* \vdash\Vdash^{\tau} \mathsf{WKL}_0^*$.
2. $\mathsf{RCA}_0 \vdash\Vdash^{\tau} \mathsf{WKL}_0$.
3. *τ is polynomially Γ_{STY}-reflecting over* RCA_0^*.

Corollary

1. RCA_0^* *polynomially simulates* WKL_0^* *with respect to Γ_{STY}-sentences.*
2. RCA_0 *polynomially simulates* WKL_0 *with respect to Γ_{STY}-sentences.*

Thank you!

- J. Avigad, Forcing in proof theory. Bull. Symbolic Logic 10 (2004), no. 3, 305-333.
- J. Avigad, Formalizing forcing arguments in subsystems of second-order arithmetic. Ann. Pure Appl. Logic 82 (1996), no. 2, 165-191.
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