Fixed Points meet Löb's Rule

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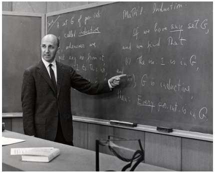
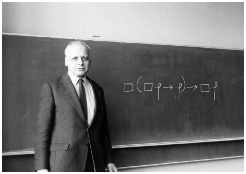

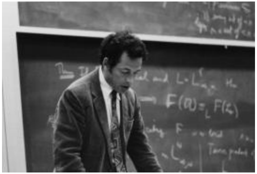

Proof Theory Virtual Seminar, November 18, 2020

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Overview

Feferman's G2 plus Examples

Uniform Semi-numerability

The Henkin Calculus

The μ-Calculus

Universiteit Utrecht

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Feferman's G2 plus Examples

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Feferman's G2

We fix an arithmetisation "proof(X, x, y)" for "x is a proof of y from axioms in X". Here X is a unary predicate symbol. We write $\mathrm{prov}_\alpha(y)$ for $\exists x\, \mathrm{proof}(\alpha, x, y)$.

Theorem ($\pm$ Feferman 1960)

Consider any consistent RE theory T and an interpretation K of $\mathrm{EA} + \mathrm{B}\Sigma_1$ in T. Suppose σ is Σ_1 and σ^K semi-numerates the axioms of T in T. Then, we have $T \nvdash (\mathrm{con}(\sigma))^K$.

One can omit $\mathrm{B}\Sigma_1$, but then we need a modification of the proof.

Limitations

- We have G2 for oracle provability, the provability notion associated with ω-consistency, cut-free EA-provability over EA, etcetera.
- EA $+ \mathrm{B}\Sigma_1$ seems far too strong to be a convincing base theory.
- The role of the very specific formula-class Σ_1.

We provide a more general Feferman-style result that works for certain predicates of the form prov_α that do not satisfy the Löb conditions.

The (non-)Role of the Löb Conditions

Feferman's *proof* employs the Löb Conditions for prov^K.

There is a Σ^0_1-numeration σ of the axioms of EA in EA such that:

- $\mathrm{EA} \vdash \exists x\, \forall y \in \sigma\; y < x$.
- $\mathrm{EA} \nvdash \Box_\sigma \Diamond_\sigma \top \leftrightarrow \Box_\sigma \bot$.
- $\mathrm{EA} \vdash G \leftrightarrow \Diamond_\sigma \Box_\sigma \bot$, for any G with $\mathrm{EA} \vdash G \leftrightarrow \neg \Box_\sigma G$.

So the Löb Conditions fail for EA. However, the *result*, G2 for Σ_1-semi-numerations, does hold —as follows from result below.

Numerability is not Sufficient

An Example due to Feferman.

Let π be a standard predicate representing the axioms of PA. Let $\pi_x(y) :\leftrightarrow \pi(y) \wedge y \leq x$.

We define $\pi^\star(y) :\leftrightarrow \pi(y) \wedge \mathrm{con}(\pi_y)$. Note that $\pi^\star$ is Π^0_1.

$\pi^\star$ numerates the axioms of PA in PA, but we *do* have $\mathrm{PA} \vdash \mathrm{con}(\pi^\star)$.

To verify in PA that the first k axioms are indeed axioms we need axioms enumerated after stage k.

Thus, the restriction to Σ_1 is needed in Feferman's Theorem.

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What goes wrong in Feferman's Example is *all* that goes wrong.

We fix $\mathsf{proof}(X, x, y)$.

Consider a consistent theory U. Let X be the set of (Gödel numbers of) axioms of U. There are no constraints on the complexity of X. Let U_k be axiomatised by X_k, the elements of X that are $\leq k$.

Suppose $N : \mathsf{S}^1_2 \lhd U$.

A U-formula ξ *uniformely semi-numerates* X (w.r.t. N) iff, for every n, there is an $m \geq n$, such that U_m proves $\xi(\underline{i})$, for each $i \in X_m$.

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A General Version of G2

Theorem

Suppose U is consistent and ξ uniformly semi-numerates the axioms X of U (w.r.t. N). Then, $U \nvdash \mathrm{con}^N[\xi]$.

The square brackets emphasise that ξ is not supposed to be relativised to N. Let B be a single sentence that axiomatises S^1_2.

Proof.

Suppose $U \vdash \mathrm{con}^N[\xi]$. Then, for some k, $U_k \vdash \mathrm{B}^N \wedge \mathrm{con}^N[\xi]$ and ξ semi-numerates X_k in U_k. Let $\beta := \bigvee_{A \in X_k}(x = \ulcorner A \urcorner)$. We find $U_k \vdash (\mathrm{B} \wedge \mathrm{con}(\beta))^N$. This contradicts a standard version of G2. ❑

$\mathrm{prov}^N_{[\xi]}$ need not to satisfy the Löb Conditions. Yet the Löb Condtions are used in the proof.

Löb's Rule

Since, uniformity can be easily lifted to finite extensions, we have:

Theorem

Suppose ξ uniformly semi-numerates the axioms of U (w.r.t. N) and $U \vdash \mathrm{prov}^{N}_{[\xi]}(\ulcorner A \urcorner) \to A$. Then $U \vdash A$.

Overview

Feferman's G2 plus Examples

Uniform Semi-numerability

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The μ-Calculus

The Henkin Calculus

We work in the language of modal logic extended with a fixed-point operator $\digamma p.\Box\varphi$. HC has the following axioms and rules.

- The axioms and rules of K,
- Löb' rule: if $\vdash \Box\varphi \to \varphi$, then $\vdash \varphi$.
- If ψ results from φ by renaming bound variables, then $\vdash \varphi \leftrightarrow \psi$.
- If $\digamma p.\Box\varphi$ is substitutable for p in φ, then $\vdash \digamma p.\Box\varphi \leftrightarrow \Box\varphi[p : \digamma p.\Box\varphi]$.

Perspective

The Henkin Calculus has standard syntax here. The disadvantage is that one has to get the details for variable-binding right —as is witnessed by the presence of the α-equivalence axiom.

One gets a sense that this material is about circularity rather than binding. A treatment using syntax on possibly cyclic graphs seems to represent what is going on much better. Such a treatment would be co-inductive. The disadvantage is discontinuity with conventional treatments.

The disadvantage of the second approach can, hopefully, be overcome by metatheorems linking the conventional treatment to the co-inductive one.

Circular Henkin Calculus

This is what the Henkin Calculus looks like on directed possibly circular graphs. Note that F is not in the language here.

We demand that in a graph that represents a formula every cycle contains a vertex labeled with a box. This is the guarding condition.

- The Axioms and Rules of K.
- Löb's rule: if $\vdash \Box\varphi \to \varphi$, then $\vdash \varphi$.
- If φ and ψ are bisimilar then $\vdash \varphi \leftrightarrow \psi$.

The Grullet Modality

Back to the ordinary syntax.

We define:

- $\Box^{\bullet}\varphi := \digamma p.\Box(\varphi \wedge p)$, where p is not free in φ.

We have:

- $\mathsf{HC} \vdash \Box^{\bullet}\varphi \leftrightarrow \Box(\varphi \wedge \Box^{\bullet}\varphi)$.
- HC verifies Löb's Logic for $\Box^{\bullet}$.
- Suppose φ is modalised in p, then
  $\mathsf{HC} \vdash (\boxdot^{\bullet}(p \leftrightarrow \varphi) \wedge \boxdot^{\bullet}(q \leftrightarrow \varphi[p:q])) \to (p \leftrightarrow q)$.
  A version of the De Jongh-Sambin-Bernardi Theorem

Gödel and Henkin sentences are unique.

Multiple Fixed Points 1

Consider a system of equations $\mathcal{E}$ given by:

$$(p_0 \leftrightarrow \varphi_0), \ldots, (p_{n-1} \leftrightarrow \varphi_{n-1}),$$

We assign a directed graph $\mathrm{G}_{\mathcal{E}}$ to $\mathcal{E}$ with as nodes the variables p_i, for $i < n$. We have an arrow from p_i to p_j iff there is an unguarded free occurrence of p_j in φ_i.

$\mathcal{E}$ is *guarded* iff $\mathrm{G}_{\mathcal{E}}$ is acyclic.

If $\mathcal{E}$ is guarded, it has a unique solution. In this solution the free variables are those of the φ_i minus the p_j.

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Multiple Fixed Points 2

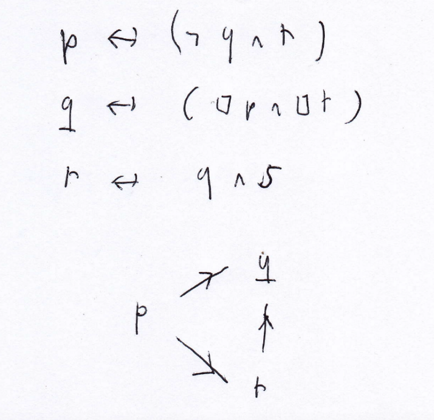

Figure: The associated graph

Reduction

Suppose φ is modalised in p. We can find a $\widetilde{\varphi}$ and fresh $q_0, \ldots, q_{n-1}$, such that p does not occur in $\widetilde{\varphi}$ and

$$\mathrm{HC} \vdash \varphi \leftrightarrow \widetilde{\varphi}[q_0 : \Box\psi_0, \ldots, q_{n-1} : \Box\psi_{n-1}].$$

By solving the equations

$$\mathcal{E} : \ p \leftrightarrow \widetilde{\varphi}, \ q_0 \leftrightarrow \Box\psi_0, \ldots, \ q_{n-1} \leftrightarrow \Box\psi_{n-1}.$$

we see that φ has a unique fixed point.

The Extended Henkin Calculus

Using the reduction result we can show that the Henkin Calculus is synonymous with its extended variant where we have fixed points for modalised formulas:

- we have $\digamma p.\varphi$ in case p only occurs in the scope of a box.

The axioms for the extended calculus are analogous.

On the circular syntax the difference between both versions disappears.

Arithmetical Interpretation

Suppose ξ uniformly semi-numerates the axioms of U w.r.t N. Then, HC is arithmetically valid in U for $\mathrm{prov}^{N}_{[\xi]}$.

The precise interpretation of the fixed-point operator and the proof of soundness take some doing.

Completeness

HC and extended HC both have a Kripke model completeness theorem in finite dags and in finite trees.

Is the provability logic of all uniformely semi-numerable axiom sets precisely HC? If so, is there a pair U, α, where this logic is assumed?

It is a win-win situation: how cool would it be to find an extra principle.

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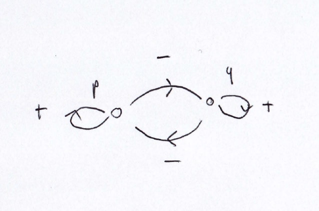
Our version of the μ-calculus consists of K plus fixed points for formulas in which the fixed-point-variable occurs positively. We write μp for the fixed-point operator reflecting our intention to look at minimal fixed points. We have the following axiom expressing minimality:

- $\vdash \varphi[p : \alpha] \to \alpha \quad \Rightarrow \vdash \mu p.\varphi \to \alpha$.

The well-founded part of μ is $\mu + \mathsf{H}$, where $\mathsf{H} := \mu p.\Box p$.

$\mu + H$ is synonymous with HC.

Well-Foundedness beats Negativity 1



Consider φ modalised in p. Let φ_0 be the result of replacing all *negative* occurrences of p in φ by a fresh q. Let φ_1 be the result of replacing all *positive* occurrences of p in φ by q.

A solution lemma tells us that the equations $p \leftrightarrow \varphi_0$, $q \leftrightarrow \varphi_1$ in μ can be solved. Let the solutions be α_0 and α_1.

Well-Foundedness beats Negativity 2

In $\mu + \mathsf{H}$, we have uniqueness of modalised fixed points for systems of equations and, hence $\mu + \mathsf{H} \vdash \alpha_0 \leftrightarrow \alpha_1$.

Ergo $\mu + \mathsf{H} \vdash \alpha_0 \leftrightarrow \varphi_0[p : \alpha_0, q : \alpha_0]$, so α_0 is a fixed point of φ.

Thank You