Comparing Rules for Identity in sequent systems & natural deduction

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PROOF THEORY VIRTUAL SEMINAR \cdot 21 APRIL 2021

https://consequently.org/presentation/2021/comparing-identity-rules

To explore different rules for an identity predicate in natural deduction and the sequent calculus.

Sequent Calculus & Natural Deduction

Defining Rules

Defining Rules for Identity

Identity Axioms

SEQUENT CALCULUS ビ NATURAL

DEDUCTION

Intuitionistic Proofs & Derivations

A derivation of $X \succ A$ builds a proof from X to A.

An Example

$$\frac{\frac{q \succ q}{q, \neg q \succ} \neg L}{\frac{q, \neg q \succ}{q, \neg q \succ} K} \xrightarrow{r \succ r} \vee L}$$

$$\frac{\frac{p \succ p}{q \lor r, \neg q \succ r} \land L}{\frac{q \lor r, \neg q \succ r}{p \rightarrow (q \lor r), p, \neg q \succ r} \rightarrow L}$$

$$\frac{p \rightarrow (q \lor r), p \land \neg q \succ r}{p \rightarrow (q \lor r) \succ (p \land \neg q) \rightarrow r} \rightarrow R$$
SEQUENT CALCULUS
$$\frac{p \rightarrow (q \lor r) \xrightarrow{p} e^{-K} \xrightarrow{r} e^{-K} [q]^{2}}{\frac{\neg q}{r} \land e^{-K} [q]^{1}} \neg e^{-K}$$

$$\frac{q \lor r}{q \lor r} \rightarrow E \xrightarrow{r} e^{-K} \xrightarrow{r} e^{-K} [r]^{1} \lor e^{-K}$$

$$\frac{q \lor r}{(p \land \neg q) \rightarrow r} \rightarrow L^{2}$$

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The one proof can be built in different ways

$$\frac{\frac{q \lor q}{q, \neg q \succ} \neg L}{\frac{p \succ p}{p \to (q \lor r), p \succ q \lor r} \to L} \qquad \frac{\frac{q \lor q}{q, \neg q \succ} \neg L}{\frac{q, \neg q \succ}{R} K} \qquad r \succ r} \lor L$$

$$\frac{\frac{p \to (q \lor r), p \succ q \lor r}{p \to (q \lor r), p, \neg q \succ r} \land L}{\frac{p \to (q \lor r), p, \neg q \succ r}{p \to (q \lor r), p \land \neg q \succ r} \to R} \qquad \text{SEQUENT CALCULUS}$$

$$\frac{p \to (q \lor r) \qquad \frac{[p \land \neg q]^2}{p \to (q \lor r) \succ (p \land \neg q) \to r} \to R}{\frac{q \lor r}{p \to (q \lor r) \succ (p \land \neg q) \to r} \to R} \qquad \text{SEQUENT CALCULUS}$$

$$\frac{p \to (q \lor r) \qquad \frac{[p \land \neg q]^2}{p \to E} \land E \qquad \frac{[p \land \neg q]^2}{\neg q \land E} \land E \qquad [q]^1}{\frac{p}{r} K} \qquad (r]^1$$

$$\frac{q \lor r}{(p \land \neg q) \to r} \to I^2$$

a > a

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Classical derivations build . . . what?

$$\frac{p \succ p}{\underset{r}{\succ} \neg p, p} \neg_{R}}{\underset{r}{\succ} \neg p, \nabla p} \lor_{R}} \qquad \qquad \frac{p \succ p, q}{\underset{r}{p, p \to q}} \rightarrow_{R}}{\frac{p \succ p, p \to q}{(p \to q) \to p \succ p, p}} \rightarrow_{L}} w$$

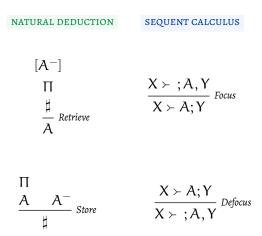
From $P_1, P_2, P_3 \succ C_1, C_2, C_3$ to $P_1, P_2, P_3 \succ C_1; C_2, C_3$

A focussed sequent has the shape $X \succ C$; Y where C is either a formula or is *empty*, and X and Y are finite multisets of formulas.

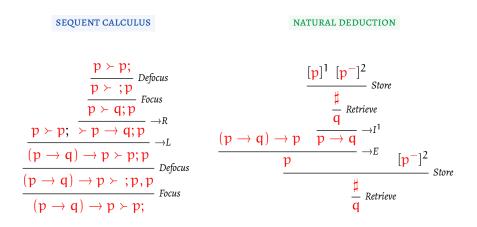
(The empty case corresponds to a proof of a *contradiction*.)

A proof for P_1 , P_2 , $P_3 \succ C_1$; C_2 , C_3 is a proof of C_1 from the context P_1 , P_2 , P_3 (positive) and C_2^- , C_3^- (negative).

Focus, Defocus; Retrieve and Store



Derivations with focus build proofs with alternatives



That's Classical logic

Adding the Store/Retrieve rules to Gentzen–Prawitz Natural Deduction gives you a well-behaved, normalising natural deduction system for classical logic.

(It's basically Michel Parigot's $\lambda\mu$ calculus.)

Interpreting Sequents with Focus

 $X \succ$; Y — a *refutation* of asserting X and denying Y.

I'll pass freely between sequent derivations and natural deduction proofs with alternatives.

DEFINING RULES

What makes rules well behaved?

$$\frac{A}{A \land B} \land I \qquad \frac{A \land B}{A} \land E \qquad \frac{A \land B}{B} \land E$$
$$\frac{A}{A \ tonk B} \ tonkI \qquad \frac{A \ tonk B}{B} \ tonkE$$

Analytic Validity

ANALYSIS

THE RUNABOUT INFERENCE-TICKET

By A. N. PRIOR

T is sometimes alleged that there are inferences whose validity arises solely from the meanings of certain expressions occurring in them. The precise technicalities employed are not important, but let us say that such inferences, if any such there be, are analytically valid.

One sort of inference which is sometimes said to be in this sense analytically valid is the passage from a conjunction to either of its conjuncts, e.g., the inference 'Grass is green and the sky is blue, therefore grass is green '. The validity of this inference is said to arise solely from the meaning of the word ' and '. For if we are asked what is the meaning of the word ' and ', at least in the purely conjunctive sense (as opposed to, e.g., its colloquial use to mean ' and then '), the answer is said to be *completely* given by saying that (i) from any pair of statements P and Q we can infer the statement formed by joining P to Q by ' and ' (which statement we hereafter describe as ' the statement P-and-Q') that (ii)

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One option . . .

One way to be analytically valid is to be a *definition*

... but $\wedge I$ and $\wedge E$ don't look much like *definitions*.

Invertible rules look more like definitions.

$$\frac{X,A,B\succ Z}{\overline{X,A\wedge B\succ Z}} \wedge \textit{Df} \qquad \frac{X,A\succ B;Y}{\overline{X\succ A\rightarrow B};Y} \rightarrow \textit{Df} \qquad \frac{X,A\succ ;Y}{\overline{X\succ \neg A};Y} \neg \textit{Df}$$

They charaterise *one* aspect of the behaviour of the introduced concept (positively or negatively). The structural rules settle the rest.

Defining rules define

They are conservative and uniquely defining.

From Defining Rules to Left/Right Rules ...

$$\frac{X, A, B \succ Z}{X, A \land B \succ Z} \land \textit{Df} \downarrow \qquad \frac{X, A, B \succ Z}{X, A \land B \succ Z} \land \textit{L}$$

 $\wedge L$ is one half of $\wedge Df$

$$\frac{X \succ A; Y}{X, X' \succ A \land B; Y, Y'} \frac{\overline{A \land B \succ A \land B;}^{Id} \land Df^{\uparrow}}{A, B \succ A \land B;}_{Cut} \qquad \qquad \frac{X \succ A; Y \quad X' \succ B; Y'}{X, X' \succ A \land B; Y, Y'} \land_{R}$$

 $\wedge R$ is formed from the other half, using *Id* and *Cut*.

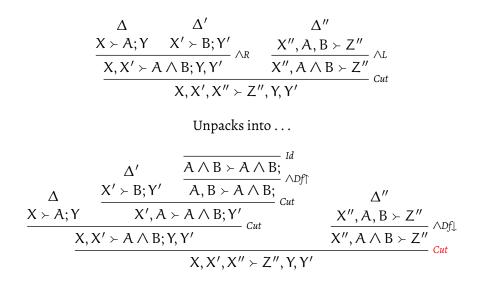
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... and back

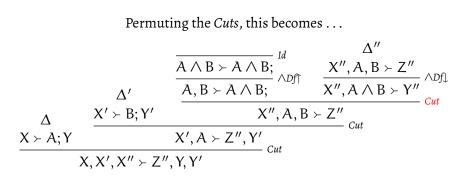
$$\frac{\overline{A \succ A;}^{Id} \quad \overline{B \succ B;}^{Id}}{\underline{A, B \succ A \land B;}^{\land R}} \stackrel{Id}{\land R} \underbrace{X, A \land B \succ Z}_{Cut} \qquad \frac{X, A \land B \succ Z}{X, A, B \succ Z} \land Df$$

We can recover $\wedge Df \uparrow$ from $\wedge R$, given *Id* and *Cut*, and $\wedge Df \downarrow$ is $\wedge L$.

L/R rules given in this way admit elimination of principal Cuts



L/R rules given in this way admit elimination of principal Cuts

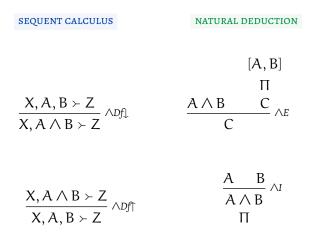


... which (since the $Id/Df\uparrow/Df\downarrow/Cut$ detour is redundant) simplifies to:

$$\frac{\Delta}{\begin{array}{c}X \succ A; Y \\ \hline X, X', X'' \succ Z'', Y \\ \hline X, X', X'' \succ Z'', Y' \\ \hline \end{array}} \begin{array}{c}\Delta' & \Delta'' \\ X'', A, B \succ Z'' \\ X'', A, B \succ Z'' \\ Cut \\ Cut \\ Cut \\ \end{array}} Cut$$

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$\wedge Df$ in Natural Deduction



Conservativity and Unique Definability

Cut Elimination and the *Subformula Property* for *rules other than Cut* gives *Conservative Extension*.

The shape of the defining rules gives Uniqueness.

Defining Rules and Generality

$$\frac{X \succ A(n); Y}{\overline{X \succ \forall x A(x); Y}} \forall Df$$

n is absent from the lower sequent, and it must be *inferentially general*.

Specification as a Rule

$$\frac{\forall \mathbf{x} \mathsf{F} \mathbf{x} \succ \forall \mathbf{x} \mathsf{F} \mathbf{x};}{\forall \mathbf{x} \mathsf{F} \mathbf{x} \succ \mathsf{F} \mathbf{n};} \underbrace{\forall \mathsf{x} \mathsf{F} \mathbf{x} \succ \mathsf{F} \mathbf{n};}_{\forall \mathbf{x} \mathsf{F} \mathbf{x} \succ \mathsf{F} \mathsf{t};} s_{pec_{\mathsf{t}}^{n}}$$

$$\frac{X \succ Z}{X[n/t] \succ Z[n/t]} \mathit{Spec}^n_t$$

The Status of Spec

Spec, like *Id* and *Cut*, are primitive rules in the system with *Df* rules.

Spec is admissible (*height preserving* admissible, in fact) as are *Id* (for complex formulas) and *Cut* in the system with *L*/*R* rules.

DEFINING RULES FOR IDENTITY

Identity and Harmony

Identity and harmony

Stephen Read

1. Harmony

The inferentialist account of logic says that the meaning of a logical operator is given by the rules for its application. Prior (1960-61) showed that a simple and straightforward interpretation of this account of logicality reduces to absurdity. For if 'tonk' has the meaning given by the rules Prior proposed for it, contradiction follows. Accordingly, a more subtle interpretation of inferentialism is needed. Such a proposal was put forward initially by Gentzen (1934) and elaborated by, e.g., Prawitz (1977). The meaning of a logical expression is given by the rules for the assertion of statements containing that expression (as designated component); these are its introduction-rules. The meaning so given justifies further rules for drawing inferences from such assertions; these are its eliminationrules:

The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequence of these definitions. (Gentzen 1934: 80)

For example, if the only ground for assertion of 'p tonk q ' is given by Prior's rule:

$$\frac{p}{p \operatorname{tonk} q} \operatorname{tonk-l}$$

then Prior mis-stated the elimination-rule. It should read

$$\frac{p \operatorname{tonk} q \quad r}{r} \operatorname{tonk-E}_{r}$$



A Defining Rule for Identity

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y} = Df$$

(Here, F is *inferentially general*, and absent from the lower sequent.)

Denying a = b has the same significance as taking there to be *some* feature F that holds of a but not b, or *vice versa*.

Or equivalently, to prove that a = b, prove Fb from the assumption Fa (and vice versa), where the predicate F is arbitrary.

Identity is a kind of *indistinguishability*.

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$= Df \uparrow$ in Natural Deduction

SEQUENT CALCULUS

$$\frac{X \succ a = b; Y}{X, Fa \succ Fb; Y} \stackrel{=Df\uparrow_1}{\underset{X, Pa \succ Pb; Y}{\longrightarrow}} \frac{X \succ a = b; Y}{\underset{P}{Spec_P^F}} = \frac{Z \succ a = b; Y}{X, Fb \succ Fa; Y} \stackrel{=Df\uparrow_2}{\underset{Spec_P^F}{\longrightarrow}}$$

NATURAL DEDUCTION

$$\frac{\Pi}{\frac{a=b}{FbPb}} = E_1 \qquad \frac{\Pi}{\frac{a=b}{FaPa}} = E_2$$

An Example Derivation

$$\frac{X, Fa \succ Fb; Y \quad X, Fb \succ Fa; Y}{X \succ a = b; Y} = Df \qquad \frac{X \succ A[x/a]; Y}{X \succ (\lambda x.A)a; Y} \lambda Df \qquad \frac{X \succ Z}{X[F/P] \succ Z[F/P]} Spec_{P}^{F}$$

$$\frac{Fa \succ Fa; \quad Fa \succ Fa;}{\underset{\succ}{\succ a = a;}{\rightarrow a = a;}{\lambda Df_{\downarrow}}} = Df_{\downarrow} \qquad \frac{a = b \succ a = b;}{a = b, Fa \succ Fb;} = Df_{\uparrow} \qquad \frac{a = b, (\lambda x. x = a)a;}{a = b, (\lambda x. x = a)a \succ (\lambda x. x = a)b;} \\ \frac{a = b \succ (\lambda x. x = a)b;}{a = b \succ b = a;} \lambda Df_{\uparrow} \qquad Cut$$

From Defining Rules to L/R rules: $=Df \downarrow$ is =R

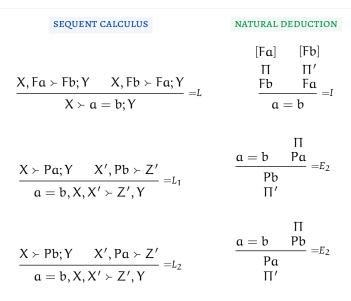
$$\frac{X, Fa \succ Fb; Y \quad X, Fb \succ Fa; Y}{X \succ a = b; Y} = Df \downarrow$$

Deriving = L rules

$$\frac{a = b \succ a = b;}{\frac{a = b, Fa \succ FbFb \succ Fa;}{a = b, Fa \succ FbFb \succ Fa;}} \stackrel{=Df\uparrow}{\underset{Cut}{a = b, X \succ PbPa; Y}} \frac{A = b, X \succ PbPb \succ Pa;}{a = b, X, X' \succ Z', Y} Cut$$

$$\frac{X \succ Pa; Y \quad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y} = L_1 \quad \frac{X \succ Pb; Y \quad X', Pa \succ Z'}{a = b, X, X' \succ Z', Y} = L_2$$

Comparing L/R rules and I/E rules



Our Symmetry Derivation in Natural Deduction

$$\frac{\begin{bmatrix} Fa \end{bmatrix} & \begin{bmatrix} Fa \end{bmatrix}}{a = a} \stackrel{=I}{\lambda I} \\ \frac{(\lambda x.x = a)a}{\lambda I} a = b}{\frac{(\lambda x.x = a)b}{b = a}} \stackrel{=E}{\lambda E}$$

With these Left/Right Rules . . .

Spec is height-preserving admissible.

We can eliminate *Cut*, as usual.

But eliminating Cut hardly seems worth it!

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y} =_{\mathbb{R}}$$

$$\frac{X \succ Pa; Y \quad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y} = L_1 \quad \frac{X \succ Pb; Y \quad X', Pa \succ Z'}{a = b, X, X' \succ Z', Y} = L_2$$

Each rule breaks the subformula property. =R might be excusable (by analogy with $\forall R / \exists L$), but in =L, P can be *any* predicate, primitive or complex.

For analytic rules, we must look elsewhere.

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IDENTITY AXIOMS

The Power of Reflexivity

$$\frac{X, Fa \succ Fb; Y \quad X, Fb \succ Fa; Y}{X \succ a = b; Y} = \mathbb{R}$$

=R says that we have a = b if we can transport a-features to b (and *vice versa*).

So we are in a position to transport a-features to b, and if we already knew that *being identical to* a was an a-feature, then that's enough show that a is identical to b.

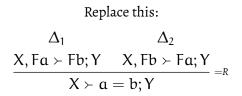
From Rules to Axioms: from =R to Refl

$$\overline{\succ a = a}$$
; Refl

$$\frac{Fa \succ Fa; \quad Fa \succ Fa;}{\succ a = a;} = \mathbb{R}$$

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And *Refl* is enough to recover = R



With this:

$$\frac{\overbrace{\succ a = a;}^{Refl}}{\overbrace{\succ \lambda x.(a = x)a;}^{\lambda R}} \begin{array}{c} \Delta_{1}[F/\lambda x.(a = x)] \\ X,\lambda x.(a = x)a; \\ X,\lambda x.(a = x)a \succ \lambda x.(a = x)b; Y \\ \hline \\ X \succ \lambda x.(a = x)b; Y \\ \hline \\ X \succ a = b; Y \end{array} _{\lambda R} Cut$$

The Problem with =L

$$\frac{X \succ Pa; Y \quad X', Pb \succ Z'}{a = b, X, X' \succ Z', Y} = L_1 \quad \frac{X \succ Pb; Y \quad X', Pa \succ Z'}{a = b, X, X' \succ Z', Y} = L_2$$

This looks just like a *Cut* on Pa/Pb, at the cost of granting a = b.

From =*L* to =*L.ax* and back.

$$\overline{\mathbf{a} = \mathbf{b}, \mathbf{P}\mathbf{a} \succ \mathbf{P}\mathbf{b};} \stackrel{=L.ax_1}{=} \overline{\mathbf{a} = \mathbf{b}, \mathbf{P}\mathbf{b} \succ \mathbf{P}\mathbf{a};} \stackrel{=L.ax_2}{=}$$

$$\frac{Pa \succ Pa; \quad Pb \succ Pb;}{a = b, Pa \succ Pb;} = L_1 \qquad \frac{Pb \succ Pb; \quad Pa \succ Pa;}{a = b, Pb \succ Pa;} = L_2$$

$$\frac{X \succ Pa; Y}{\frac{a = b, Pa \succ Pb;}{a = b, X \succ Pb; Y}} \xrightarrow{eL.ax_1} Cut \\ \frac{a = b, X \succ Pb; Y}{a = b, X, X' \succ Z', Y} Cut$$

$$\frac{X \succ Pb; Y \qquad \overline{a = b, Pb \succ Pa;} \stackrel{=L.ax_2}{} \\ \frac{a = b, X \succ Pa; Y \qquad Cut}{a = b, X, X' \succ Z', Y} Cut$$

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We can restrict =*L.ax* to *primitive* predicates

$$\frac{\overline{a = b, Pa \succ Pb};}{a = b, Pa \land Qa \succ Pb}; \land L} \qquad \frac{\overline{a = b, Qa \succ Qb};}{a = b, Pa \land Qa \succ Qb}; \land L} \land L$$

$$\frac{\overline{a = b, Pa \land Qa \succ Pb};}{a = b, Pa \land Qa \succ Pa \land Qb}; \land R$$

$$\frac{a = b, Ax.(Px \land Qx)a \succ Ax.(Px \land Qx)b}; \land R$$

We can restrict =*L.ax* to *primitive* predicates

$$\frac{\overline{a = b, Pb \succ Pa;}^{=L.ax_{2}}}{\frac{a = b, Pb, \neg Pa \succ;}{a = b, \neg Pa \succ \neg Pb;}}^{\neg L} \overline{a = b, \neg Pa \succ \neg Pb;}^{\neg R}}_{\lambda = b, \lambda x. (\neg Px) a \succ \lambda x. (\neg Px) b;}$$

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We can restrict =*L.ax* to *primitive* predicates

$$\frac{\overline{a = b, Pac \succ Pbc;}^{=L.ax_{2}}}{a = b, \forall y Pay \succ Pbc;}^{\neg L}} \frac{\neg L}{a = b, \forall y Pay \succ Pbc;}^{\neg R}}_{a = b, \lambda x.(\forall y Pxy) a \succ \lambda x.(\forall y Pxy) b;}^{\neg R}$$

Identity axioms in Natural Deduction

SEQUENT CALCULUS NATURAL DEDUCTION $\overline{} = a;$ Refl a = a $\frac{a=b Pa}{Pb} = E$ $\overline{a = b, Pa \succ Pb}$; L.ax₁ $\frac{a=b}{E} = E$ $\overline{a = b, Pb \succ Pa}$; L.ax₂ Pa

Now eliminate Cut

Now that identity is given by axioms, *Cut* elimination proceeds largely like the system *without* identity.

 $\frac{\overline{a = b, Pa \succ Pb;} \stackrel{=L.ax_1}{} \frac{}{c = b, Pb \succ Pc;} \stackrel{=L.ax_2}{} a = b, c = b, Pa \succ Pc;} cut$

becomes

$$\frac{1}{a = b, c = b, Pa \succ Pc;} = L??$$

Now eliminate Cut

It suffices to close the axioms under Cut.

$$\frac{1}{I_b^a, Pa \succ Pb;} = L.ax^*$$

Where I^a_b is any multiset of identities *linking* a *to* b, and P is any primitive predicate.

(a) The *empty* multiset links a to a.

(b) a = b links a to b and b to a.

(c) If X links a to b and Y links b to c then X, Y links a to c.

(We can leave 'Pa' out if it is a = a.)

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Different Sequent Systems

- \blacktriangleright =Df+Cut+Spec
 - It's easy to show that =Df is uniquely defining.
- \blacktriangleright =L/R + Cut + Spec
 - Straightforward translation between =Df and =L/R.
- \blacktriangleright =L/R + Cut
 - Since *Spec* is height-preserving admissible.
- \blacktriangleright =L/R
 - =*L*/*R* rules don't have the subformula property.
- \blacktriangleright =L.ax + Refl + Cut
 - Easy translation between =L/R and =L.ax + Refl, using Cut.
- \blacktriangleright =L.ax* + Refl

- =L.ax^{*} + Refl are analytic and conservatively extending.

Kinds of Identity Rules

$$\frac{X, Fa \succ Fb; Y \qquad X, Fb \succ Fa; Y}{X \succ a = b; Y} = Df$$

$$\frac{1}{r} = a; \quad \text{Refl} \quad \overline{I_b^a, Pa \succ Pb}; \quad \text{L.ax}_*$$

- =Df defines identity by giving conditions under which a = b may be proved. We're in a position to prove a = b iff we're in a position to transfer Fa to Fb (and back) for arbitrary F.
- ▶ Refl and =*L*.ax_{*} are semantic constraints connecting primitive predicates.
- These two characterisations are *equivalent* as far as derivability goes.

THANK YOU!

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Thank you!

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