Far beyond Goodman's Theorem?

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includes joint work with Emanuele Frittaion

To **AC**, or not to **AC**

Zermelo 1904: $\mathbb{R}$ can be well-ordered.

Borel canvassed opinions of the most prominent French mathematicians of his generation - Hadamard, Baire, and Lebesgue.

It seems to me that the objection against it is also valid for every reasoning where one assumes an arbitrary choice made an uncountable number of times, for **such reasoning does not belong in mathematics**.
Borel, *Mathematische Annalen* 1905.

Use **AC**, and remove **AC**

Theorem (Gödel 1938, 1940) **ZFC** + **GCH** is conservative over **ZF** for arithmetic sentences.

Theorem (Shoenfield 1961) **ZFC** + **GCH** is conservative over **ZF** for Π^1_4 sentences.

Theorem (Goodman 1976, 1978) $\mathbf{HA}^\omega + \mathbf{AC}_{FT}$ is conservative over **HA** for arithmetic sentences.

Here $\mathbf{HA}^\omega$ denotes Heyting arithmetic in all finite types with $\mathbf{AC}_{FT}$ standing for the collection of all higher type versions $\mathbf{AC}_{\sigma\tau}$ of the axiom of choice with σ, τ arbitrary finite types.

Independence and conservativity results in classical set theory

1. Inner Models
   The **Constructible Hierarchy**, **L** (Gödel)
2. **Forcing** (Cohen)
   - $\mathbb{P}$ partial order, M model of set theory, $\mathbb{P} \in M$, G filter on $\mathbb{P}$ and generic over M. $M[G]$ generic extension of M.
   - **Permutation models** for proving the independence of **AC**. Alternatively take $\mathbf{HOD}(\mathbb{P})^{M[G]}$ with suitably chosen $\mathbb{P}$ (homogeneous).

Doing the constructible hierarchy constructively

Theorem (Bob Lubarsky) $\mathbf{IZF} \vdash (\mathbf{IZF})^L$ and $\mathbf{IZF} \vdash (V = L)^L$.

Theorem (Laura Crosilla) $\mathbf{IKP} \vdash (\mathbf{IKP})^L$ and $\mathbf{IKP} \vdash (V = L)^L$.

Theorem (Richard Matthews) $\mathbf{IZF} \nvdash \forall\alpha\ \alpha \in L$.

Theorem (R.) $\mathbf{CZF} \nvdash (\mathbf{CZF})^L$.

Independence and conservativity results for intuitionistic/constructive set theories

1. **Realizability** interpretations
2. **Kripke** models
3. **Forcing** and **Heyting-valued** models
4. **Permutations** models
5. **Topological** and **Sheaf** models
6. The **formulae-as-classes** or **formulae-as-types** interpretation
7. **Categorical** models, **Topoi**, **Algebraic Set Theory**
8. **Proof-theoretic** methods

Intuitionistic Zermelo-Fraenkel set theory, **IZF**

* **Extensionality**
- **Pairing**, **Union**, **Infinity**
- **Full Separation**
- **Powerset**

\# **Collection**

$$(\forall x \in a)\,\exists y\ \varphi(x,y) \quad \rightarrow \quad \exists b\ (\forall x \in a)\,(\exists y \in b)\ \varphi(x,y)$$

* **Set Induction**

$$(IND_{\in}) \qquad \forall a\ (\forall x \in a\ \varphi(x) \rightarrow \ \varphi(a)) \rightarrow \forall a\ \varphi(a),$$

Myhill's $\mathbf{IZF}_R$:

IZF with **Replacement** instead of **Collection**

Constructive Zermelo-Fraenkel set theory, **CZF**

- * **Extensionality**
- ▶ **Pairing**, **Union**, **Infinity**
- ▶ **Bounded Separation**
- # **Subset Collection**

  For all sets A, B there exists a “sufficiently large” set of multi-valued functions from A to B.
- # **Strong Collection**

$$(\forall x \in a)\, \exists y\ \varphi(x, y) \ \rightarrow$$
$$\exists b\, [\, (\forall x \in a)\, (\exists y \in b)\ \varphi(x, y)\ \wedge\ (\forall y \in b)\, (\exists x \in a)\ \varphi(x, y)\,]$$

- * **Set Induction scheme**

Set theory with an elementary embedding

Expand the language of ordinary set theory by a unary predicate symbol M and a unary function symbol $\jmath$.
Add the following axioms:

M is transitive and $M \models$ **IZF**.

$\exists x\, x \in \jmath(x)$ and $\jmath : V \to M$ is an elementary embedding, i.e.

$$\forall x_1 \ldots \forall x_n\, [A(x_1, \ldots, x_r) \leftrightarrow A^M(j(x_1), \ldots, j(x_r))]$$

for all formulas $A(x_1, \ldots, x_r)$ of **IZF**.

Extend the axiom schemata of **IZF** to the richer language.

Intuitionistic Reinhardt set theory has the additional axioms saying that $V = M$ and

$$\exists z\, [z \text{ inaccessible set} \wedge z \in \jmath(z) \wedge \forall x \in z\, \jmath(x) = x].$$

Kleene 1945 realizability

Write $e \bullet n$ for $\{e\}(n)$. $\langle , \rangle$ is a primitive recursive and bijective pairing function on $\mathbb{N}$. Let $(e)_0 = n$ and $(e)_1 = k$ where n, k are uniquely determined by $e = \langle n, k \rangle$.

- $e \Vdash_K A$ iff A is true for atomic A.
- $e \Vdash_K A \wedge B$ iff $(e)_0 \Vdash_K A$ and $(e)_1 \Vdash_K B$
- $e \Vdash_K A \vee B$ iff $(e)_0 = 0 \ \wedge \ (e)_1 \Vdash_K A$ or $(e)_0 \neq 0 \ \wedge \ (e)_1 \Vdash_K B$
- $e \Vdash_K A \rightarrow B$ iff $\forall d \in \mathbb{N}[d \Vdash_K A \rightarrow e \bullet d \downarrow \ \wedge e \bullet d \Vdash_K B]$
- $e \Vdash_K \forall x F(x)$ iff for all $n \in \mathbb{N}$, $e \bullet n \downarrow \ \wedge e \bullet n \Vdash_K F(n)$
- $e \Vdash_K \exists x F(x)$ iff $(e)_1 \Vdash_K F((e)_0)$.

Schönfinkel algebras and PCA's

Moses Ilyich Schönfinkel: *Über die Bausteine der mathematischen Logik* (1924, talk in Göttingen 7.12.1920)

Definition. A *PCA* is a structure $(M, \cdot)$, where $\cdot$ is a partial binary operation on M, such that M has at least two elements and there are elements $\mathbf{k}$ and $\mathbf{s}$ in M such that $(\mathbf{k} \cdot x) \cdot y$ and $(\mathbf{s} \cdot x) \cdot y$ are always defined, and

(i) $(\mathbf{k} \cdot x) \cdot y = x$
(ii) $((\mathbf{s} \cdot x) \cdot y) \cdot z \simeq (x \cdot z) \cdot (y \cdot z)$,

where $\simeq$ means that the left hand side is defined iff the right hand side is defined, and if one side is defined then both sides yield the same result.

$(M, \cdot)$ is a total PCA if $a \cdot b$ is defined for all $a, b \in M$.

The theory **PCA**

The **logic** of **PCA** is assumed to be that of intuitionistic predicate logic with identity.
PCA's **non-logical axioms** are the following:

Axioms of PCA

1. $ab \simeq c_1 \ \wedge \ ab \simeq c_2 \ \rightarrow \ c_1 = c_2$.
2. $(\mathbf{k}ab) \downarrow \ \wedge \ \mathbf{k}ab \simeq a$.
3. $(\mathbf{s}ab) \downarrow \ \wedge \ \mathbf{s}abc \simeq ac(bc)$.
4. $\mathbf{k} \neq \mathbf{s}$.

Models of **PCA**

Proposition. Every *pca* can be expanded to an applicative structure. $\mathbf{PCA}^+$ is conservative over **PCA**.

- The first Kleene algebra: Turing machine application.
- The second Kleene algebra: Continuous function application in Baire space $\mathbb{N}^{\mathbb{N}}$.
- Term models.
- The graph model $\mathcal{P}(\omega)$ and its substructures.
- The Scott D_∞ models over any partial order that is complete with respect to denumerable ascending chains (ω-dcpo).
- Nonstandard models of **PA**.
- Set recursion over admissible sets.
- Recursion in a higher type functional
- α-recursion, etc.

Generic Realizability for Set Theory

This type of realizability goes back to Kreisel and Troelstra (for second order arithmetic). It was extended to intensional set theory by Friedman and Beeson. The final step to extensional set theory was taken by McCarty. This concerns the atomic case and is basically the same as for boolean valued models (forcing).

The general realizability structure

$\mathcal{A}$ will be assumed to be a fixed but arbitrary PCA whose domain is denoted by $|\mathcal{A}|$. $\mathcal{P}(X)$ stands for the power set of X.
Ordinals are transitive sets whose elements are transitive also. We use lower case Greek letters to range over ordinals.

$$(1) \qquad \mathsf{V}(\mathcal{A})_\alpha = \bigcup_{\beta\in\alpha} \mathcal{P}\big(|\mathcal{A}| \times \mathsf{V}(\mathcal{A})_\beta\big).$$

$$(2) \qquad \mathsf{V}(\mathcal{A}) = \bigcup_{\alpha} \mathsf{V}(\mathcal{A})_\alpha.$$

If $a \in \mathsf{V}(\mathcal{A})$ and $x \in a$, then x

$$x = \langle e, b \rangle$$

for some $e \in |\mathcal{A}|$ and $b \in \mathsf{V}(\mathcal{A})$.

Definition:

Let $a, b \in \mathsf{V}(\mathcal{A})$ and $e \in |\mathcal{A}|$.

$$
\begin{array}{lll}
e \Vdash \varphi \wedge \psi & \text{iff} & (e)_0 \Vdash \varphi \ \wedge \ (e)_1 \Vdash \psi \\
e \Vdash \varphi \vee \psi & \text{iff} & \big[(e)_0 = \mathbf{0} \ \wedge \ (e)_1 \Vdash \varphi\big] \\
 & & \vee \ \big[(e)_0 = \mathbf{1} \ \wedge \ (e)_1 \Vdash \psi\big] \\
e \Vdash \neg\varphi & \text{iff} & \forall f \in |\mathcal{A}| \ \neg f \Vdash \varphi \\
e \Vdash \varphi \to \psi & \text{iff} & \forall f \in |\mathcal{A}| \ \big[f \Vdash \varphi \ \to \ ef \Vdash \psi\big] \\
e \Vdash \forall x \varphi & \text{iff} & \forall c \in \mathsf{V}(\mathcal{A}) \ e \Vdash \varphi[x/c] \\
e \Vdash \exists x \varphi & \text{iff} & \exists c \in \mathsf{V}(\mathcal{A}) \ e \Vdash \varphi[x/c]
\end{array}
$$

The atomic cases

Definition:

$$e \Vdash a \in b \quad \textit{iff} \quad \exists c \left[\langle (e)_0, c \rangle \in b \ \wedge \ (e)_1 \Vdash a = c \right]$$

$$\begin{aligned} e \Vdash a = b \quad \textit{iff} \quad & \forall f, d \left[\left(\langle f, d \rangle \in a \ \rightarrow \ (e)_0 f \Vdash d \in b \right) \right. \\ & \left. \wedge \left(\langle f, d \rangle \in b \ \rightarrow \ (e)_1 f \Vdash d \in a \right) \right] \end{aligned}$$

Worlds

- $\mathsf{V}(K_1) \models$ Russian Constructivism
- $\mathsf{V}(K_2) \models$ Brouwer's Intuitionism

Finite Types

Finite types σ and their associated extensions F_σ are defined by the following clauses:

- $o \in \mathrm{FT}$ and $F_o = \omega$;
- if $\sigma, \tau \in \mathrm{FT}$, then $(\sigma)\tau \in \mathrm{FT}$ and

$$F_{(\sigma)\tau} = F_\sigma \to F_\tau = \{\text{total functions from } F_\sigma \text{ to } F_\tau\}.$$

For brevity we write $\sigma\tau$ for $(\sigma)\tau$, if the type σ is written as a single symbol. We say that $x \in F_\sigma$ has type σ.
The set FT of all finite types, the set $\{F_\sigma : \sigma \in \mathrm{FT}\}$, and the set $\mathfrak{F} = \bigcup_{\sigma \in \mathrm{FT}} F_\sigma$ all exist in **CZF**.

Axiom of Choice in Finite Types

Finite type **AC**, $\mathbf{AC}_{FT}$, consists of the formulae

$$\forall x^\sigma \, \exists y^\tau \, A(x, y) \to \exists f^{\sigma\tau} \, \forall x^\sigma \, A(x, f(x)),$$

where σ and τ are (standard) finite types.

We write $\forall x^\sigma \, B(x)$ and $\exists x^\sigma \, B(x)$ as a shorthand for $\forall x \, (x \in F_\sigma \to B(x))$ and $\exists x \, (x \in F_\sigma \wedge B(x))$ respectively.

History of Extensional Realizability

- Robin Grayson (1981) for first order arithmetic; Andrew Pitts (1981)
- Beeson (1985)
- Gordeev (1988)
- van Oosten (1997)
- Troelstra (1998)

From now on it's joint work with Emanuele Frittaion

The Extensional Realizability Structure

$\mathcal{A}$ is again an arbitrary PCA with domain $|\mathcal{A}|$. $\mathcal{P}(X)$ stands for the power set of X.

$$(3) \qquad \mathsf{V}_{ex}(\mathcal{A})_\alpha = \bigcup_{\beta\in\alpha} \mathcal{P}\big(|\mathcal{A}| \times |\mathcal{A}| \times \mathsf{V}_{ex}(\mathcal{A})_\beta\big).$$

$$(4) \qquad \mathsf{V}_{ex}(\mathcal{A}) = \bigcup_{\alpha} \mathsf{V}_{ex}(\mathcal{A})_\alpha.$$

If $x \in \mathsf{V}_{ex}(\mathcal{A})$ and $y \in x$, then

$$y = \langle e, e', z\rangle$$

for some $e, e' \in |\mathcal{A}|$ and $z \in \mathsf{V}_{ex}(\mathcal{A})$.

The intuition for $\langle e, e', y\rangle \in x$ is that e and e' are *equal* realizers for $y^{\mathcal{A}} \in x^{\mathcal{A}}$, where $x^{\mathcal{A}} = \{z^{\mathcal{A}} \colon \langle e, e', z\rangle \in x \text{ for some } e, e' \in \mathcal{A}\}$.

Extensional Generic Realizability

Define $a = b \Vdash \varphi$, where $a, b \in |\mathcal{A}|$ and φ is a formula with parameters in $\mathrm{V}_{ex}(\mathcal{A})$. Atomic cases are defined by transfinite recursion.

$$
\begin{array}{lcl}
a = b \Vdash x \in y & \Leftrightarrow & \exists z\,(\langle (a)_0, (b)_0, z\rangle \in y \wedge (a)_1 = (b)_1 \Vdash x = z) \\
a = b \Vdash x = y & \Leftrightarrow & \forall \langle c, d, z\rangle \in x\,((ac)_0 = (bd)_0 \Vdash z \in y) \text{ and} \\
 & & \forall \langle c, d, z\rangle \in y\,((ac)_1 = (bd)_1 \Vdash z \in x) \\
a = b \Vdash \varphi \wedge \psi & \Leftrightarrow & (a)_0 = (b)_0 \Vdash \varphi \wedge (a)_1 = (b)_1 \Vdash \psi \\
a = b \Vdash \varphi \vee \psi & \Leftrightarrow & (a)_0 = (b)_0 = \mathbf{0} \wedge (a)_1 = (b)_1 \Vdash \varphi \text{ or} \\
 & & (a)_0 = (b)_0 = \mathbf{1} \wedge (a)_1 = (b)_1 \Vdash \psi \\
a = b \Vdash \neg\varphi & \Leftrightarrow & \forall c, d \in A\, \neg(c = d \Vdash \varphi) \\
a = b \Vdash \varphi \rightarrow \psi & \Leftrightarrow & \forall c, d\,(c = d \Vdash \varphi \rightarrow ac = bd \Vdash \psi) \\
a = b \Vdash \forall u \in y\, \varphi(u) & \Leftrightarrow & \forall \langle c, d, x\rangle \in y\ ac = bd \Vdash \varphi(x) \\
a = b \Vdash \exists u \in y\, \varphi(u) & \Leftrightarrow & \exists x\,[\langle (a)_0, (b)_0, x\rangle \in y \wedge (a)_1 = (b)_1 \Vdash \varphi(x)] \\
a = b \Vdash \forall u\, \varphi(u) & \Leftrightarrow & \forall x \in \mathrm{V}(A)\ a = b \Vdash \varphi(x) \\
a = b \Vdash \exists u\, \varphi & \Leftrightarrow & \exists x \in \mathrm{V}(A)\ a = b \Vdash \varphi(x)
\end{array}
$$

We write $a \Vdash \varphi$ for $a = a \Vdash \varphi$, and also $a \Vdash_{\mathcal{A}} \varphi$ to highlight the underlying PCA $\mathcal{A}$.

Realizability Theorems

Theorem 1 Whenever $\mathbf{CZF} + \mathbf{AC}_{FT} \vdash \varphi$ one can effectively construct an application term t such

$$\mathbf{CZF} \vdash \forall \mathcal{A} \, [\mathcal{A} \text{ PCA} \to t^{\mathcal{A}} = t^{\mathcal{A}} \Vdash_{\mathcal{A}} \varphi]$$

Theorem 2 Whenever $\mathbf{IZF} + \mathbf{AC}_{FT} \vdash \varphi$ one can effectively construct an application term t such

$$\mathbf{IZF} \vdash \forall \mathcal{A} \, [\mathcal{A} \text{ PCA} \to t^{\mathcal{A}} = t^{\mathcal{A}} \Vdash_{\mathcal{A}} \varphi]$$

One can add many more principles, e.g. **DC**, **RDC**, Presentation Axiom, large set axioms (regular extension axiom, inaccessible, Mahlo, Reinhardt).

Goodman type theorems

I stole the title from a section in the paper *Large sets in intuitionistic set theory* by Friedman and Ščedrov from 1984.
I Let $\mathbf{CAC}_{FT}$ and $\mathbf{DC}_{FT}$ be the following schemata:

$$\forall n\, \exists y^{\tau}\, \varphi(n, y) \to \exists f^{0\tau}\, \forall n\, \varphi(n, f(n))$$
$$\forall x^{\sigma}\, \exists y^{\sigma}\, \varphi(x, y) \to \forall x^{\sigma}\, \exists f^{0\sigma}\, [f(0) = x \;\wedge\; \forall n\, \varphi(f(n), f(n+1))]$$

Theorem **IZF** plus the schemata CAC_{FT} and DC_{FT} (restricted to parameters in FT) is conservative over **IZF** for arithmetic sentences.

Proof Strategy à la Goodman/Beeson

Theorem IZF plus the schemata CAC_{FT} and DC_{FT} (restricted to parameters in FT) is conservative over **IZF** for arithmetic sentences.
They proceed as follows:

- Adjoin a new constant g to the language of **IZF**.
- Add an axiom saying that g is a partial function from ω to ω.
- Use a notion of realizability based on indices for g-oracle partial recursive functions and show that CAC_{FT} and DC_{FT} hold in the pertaining realizability model.
- For a given arithmetic statement A, define a notion of forcing $\mathbb{P}$ based on finite sequences of numbers so that
  1. $$\forall p \in \mathbb{P}\, \exists q \in \mathbb{P}\, [q \supset p \wedge q \text{ forces } ((e \Vdash A) \to A)],$$
  2. $$\forall p \in \mathbb{P}\, [(p \text{ forces } A) \text{ iff } A].$$

Stronger Goodman type theorems

With extensional realizability one gets a stronger result:

Theorem **IZF** plus the schemata AC_{FT} is conservative over **IZF** for arithmetic sentences.

What more to expect from extensional realizability?

- Finite types aren't the limit. Go to transfinite types such as

$$\prod_{\sigma \in \mathsf{T}} F_\sigma$$

  and **AC** for such types. The dependent type constructors Σ and Π are crucial in Martin-Löf type theory. Use extensional realizability to realize **AC** for the types of **MLTT** and get Goodman-style conservativity.
- Study extensional realizability for specific PCA's. E.g. K_2 goes well with continuity principles.
- Combine extensional realizability with truth to show that set theories are closed under the $\mathbf{AC}_{\sigma\tau}$-rules:

$$T \vdash \forall x^\sigma \, \exists y^\tau \, A(x, y) \quad \Rightarrow \quad T \vdash \exists z^{\sigma\tau} \, \forall x^\sigma \, A(x, z(x))$$

What more to expect from extensional realizability?

- A PCA $\mathcal{A}$ is "natural" if it has a natural representation in $V_{ex}(\mathcal{A})$. In that case $V_{ex}(\mathcal{A})$ realizes **AC** for $\mathcal{A}$. The most comon PCA's are rather small from a set-theoretic point of view. But one can basically create PCA's out of any set in the universe. This seems to be a way make **AC** true for large sets.

Thanks for listening