university of groningen

Up and Down the Lambek Calculus

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Proof Theory Virtual Seminar

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*“ Read your contracts.
Up and down,
left and right.”* LJ

1. proof theory $\hookrightarrow$ decidability & upper bounds for substructural logics

2. many extensions of FLec (essentially IMALLC) and FLew (IMALLW)

3. I will focus on the (structural) proof theoretic motifs

This talk is based on the following.

1. Extended Kripke lemma and decidability for hypersequent substructural logics. RR.
   LICS 2020.
2. Decidability and Complexity in Weakening and Contraction Hypersequent Substructural Logics.
   A. R. Balasubramanian, Timo Lang, RR.
   Accepted at LICS 2021.

Joint work with A. R. Balasubramanian (TU Munich) and Timo Lang (TU Vienna)

$\hookrightarrow$ starting point Kripke and Urquhart

Kripke's proof of decidability for FLec (1959)

Multiplicative fragment

$$\frac{}{p \Rightarrow p} \qquad \frac{X, X, Y \Rightarrow C}{X, Y \Rightarrow C}\ \text{contraction}$$

$$\frac{A, B, X \Rightarrow C}{A \cdot B, X \Rightarrow C} \qquad \frac{X \Rightarrow A \qquad Y \Rightarrow B}{X, Y \Rightarrow A \cdot B}$$

$$\frac{A, X \Rightarrow B}{X \Rightarrow A \to B} \qquad \frac{X \Rightarrow A \qquad B, Y \Rightarrow C}{A \to B, X, Y \Rightarrow C}$$

$$\frac{}{\Rightarrow 1} \qquad \frac{}{0 \Rightarrow}$$

Additive rules

$$\frac{A_i, X \Rightarrow C}{A_1 \wedge A_2, X \Rightarrow C} \qquad \frac{X \Rightarrow A \qquad X \Rightarrow B}{X \Rightarrow A \wedge B}$$

$$\frac{A, X \Rightarrow C \qquad B, X \Rightarrow C}{A \vee B, X \Rightarrow C} \qquad \frac{X \Rightarrow A_1}{X \Rightarrow A_1 \vee A_2}$$

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Multiplicative fragment

$$\frac{}{p \Rightarrow p} \qquad \frac{X, X, Y \Rightarrow C}{X, Y \Rightarrow C} \text{ contraction}$$

$$\frac{A, B, X \Rightarrow C}{A \cdot B, X \Rightarrow C} \qquad \frac{X \Rightarrow A \quad Y \Rightarrow B}{X, Y \Rightarrow A \cdot B}$$

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Additive rules

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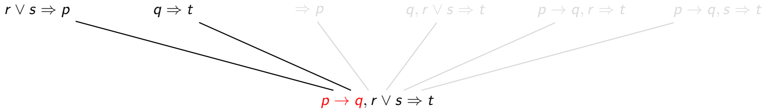
$$\frac{A, X \Rightarrow C \quad B, X \Rightarrow C}{A \vee B, X \Rightarrow C} \qquad \frac{X \Rightarrow A_1}{X \Rightarrow A_1 \vee A_2}$$

Decision problem for FLec

1. Is there a proof of $(((p \to p \cdot p) \to q) \cdot (q \to q)) \to q$?

2. Backward proof search repeatedly write all premises as a child



3. if termination then decision procedure

   F is provable iff subtree of proof search tree is a proof

4. termination here? NO: Contraction rule will be applied indefinitely

$\hookrightarrow$ annoying rules

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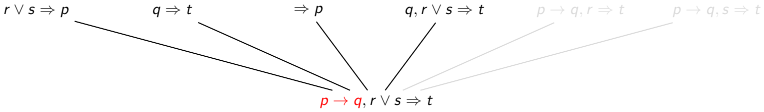

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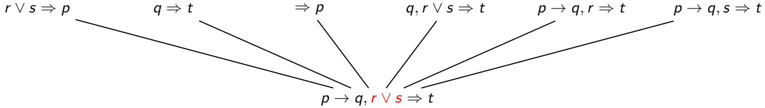

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↪ annoying rules

Structural proof theory: if there's an annoying rule. . .

Structural proof theory: if there's an annoying rule... eliminate it!

$\hookrightarrow$ absorbing contraction

Add rule variants to **absorb essential contractions** so can eliminate (c)

$$\frac{p \to q \Rightarrow p \qquad p \to q, q \Rightarrow}{\underbrace{p \to q, p \to q, p \to q}_{3} \Rightarrow} \to L$$

original rule

$$\frac{p \to q \Rightarrow p \qquad p \to q, q \Rightarrow}{\underbrace{p \to q, p \to q}_{2} \Rightarrow} \to L^1$$

variant: one implicit contraction

$$\frac{p \to q \Rightarrow p \qquad p \to q, q \Rightarrow}{p \to q \Rightarrow} \to L^2$$

variant: two implicit contractions

What happens if we need to contract 4 copies to 1?

$$\frac{\dfrac{p \to q, p \to q \Rightarrow p \qquad p \to q, q \Rightarrow}{p \to q, p \to q, p \to q, p \to q \Rightarrow}}{p \to q \Rightarrow} \text{c,c,c}$$

Above variants suffice - do additional contraction above implication rule:

$$\frac{\dfrac{p \to q, p \to q \Rightarrow p}{p \to q \Rightarrow p}\,\text{IH} \qquad p \to q, q \Rightarrow}{p \to q \Rightarrow} \to L^2$$

Lemma (hp contraction in new calculus - Curry's lemma)

If $X, X, Y \Rightarrow C$ provable then $X, Y \Rightarrow C$ provable with no greater height

↪ representing sequents

Representing sequents as n-tuples

1. A proof of F contains only its subformulas (subformula property)

2. A sequent is

$$|\mathrm{subf}(F)|\text{-tuple} \Rightarrow \text{“formula or blank”}$$

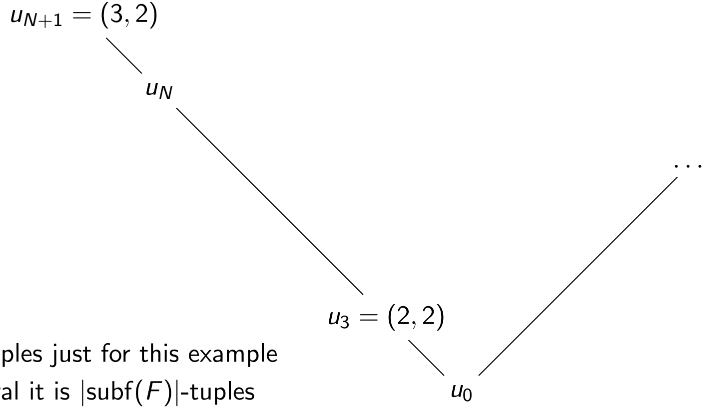
$$\text{Suppose } \mathrm{subf}(F) = \{p, q, r, r \to q\}$$

$$q, r \to q, q, p \Rightarrow r \quad \text{written as} \quad (\overset{p}{1}, \overset{q}{2}, \overset{r}{0}, \overset{r \to q}{1}) \Rightarrow r$$

3. mostly ignore the RHS since it doesn't cause any real complications

$\hookrightarrow$ define a branch termination condition

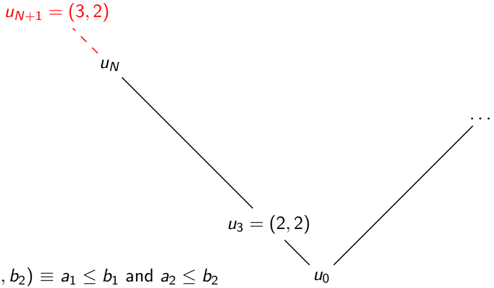
Terminating proof search tree via redundancy



using 2-tuples just for this example
in general it is $|\mathrm{subf}(F)|$-tuples

$\hookrightarrow$ how to prune tree?

Terminating proof search tree via redundancy



$(a_1, a_2) \leq (b_1, b_2) \equiv a_1 \leq b_1$ and $a_2 \leq b_2$

1. **omit** u_{N+1} **since** $u_3 \leq u_{N+1}$. Because this proof is not everywhere minimal (every subproof has minimal height - terminology in Larchey-Wendling 2018) by hp-contraction
2. (completeness) every provable formula has everywhere minimal proof
3. $(u_0, u_1, \ldots, u_N)$ is a bad sequence since $i < j$ implies $u_i \not\leq u_j$
4. $(\mathbb{N}^k, \leq)$ is a well-quasi-ordering i.e. there is no infinite bad sequence
5. finitely branching tree & no infinite branch = proof search tree finite

$\hookrightarrow$ complexity

Urquhart's tight complexity bounds (1999)

1. upper bound: what is the height of the proof-search tree? This is the dominant term.
2. No bound in general for bad sequences. After all:
3. $(1, 0), (0, 100)$ or even $(1, 0), (0, 1000)$ $\ldots$ i.e. arbitrarily large jumps
4. However no rule in FLec can jump so much from conclusion to premise

$\hookrightarrow$ controlled bad sequences

Controlled bad sequences

1. $\exists$ control function g bounding premise size in terms of conclusion
2. bad sequence $a_0, a_1, \ldots$ is (g, n)-controlled over a normed wqo $(A, \| \|, \leq_A)$ if

   $$\|a_0\| \leq n \qquad \|a_1\| \leq g(n) \qquad \|a_2\| \leq g(g(n)) \qquad \|a_k\| \leq g^k(n)$$

   and $\{a \in A \text{ s.t. } \|a\| \leq n\}$ finite for every $n \in \mathbb{N}$
3. dominant term in complexity: max length of bad sequence
4. The length function theorem expresses this length. Using this:
5. FLec decision problem is in $\mathbf{F}_\omega$ i.e. primitive recursive functions composed with a single application of an Ackermannian function
6. Controlled bad sequences: (Figueira, Figueira, Schmitz, Schnoebelen, 2011) and (Schmitz, Schnoebelen, 2011).
7. Urquhart showed that this is tight by giving matching lower bounds. Also: multiplicative fragment 2*EXPTIME*-complete (Schmitz, 2016).

$\hookrightarrow$ extending to other logics

Extending Kripke's argument to more logics

1. Kripke's argument doesn't really depend on the calculus (mainly contraction absorption)

2. So can we extend to other logics? some isolated results since 1959

3. sequent calculus meta-language too restrictive for cut-freeness

   extend meta-language to get cut-freeness i.e. different proof formalism

4. hypersequent calculus - a calculus on multisets of sequents

$$\frac{\cdots|\cdots|\cdots|X_1, X_2 \Rightarrow B \qquad \cdots|\cdots|\cdots|Y_1, Y_2 \Rightarrow C}{\cdots|\cdots|\cdots|X_1, Y_1 \Rightarrow B|Y_2, Y_2 \Rightarrow C}\ \text{com}$$

   Let HFLe denote hypersequent calculus for $\mathsf{FL_e}$

5. lots of extensions of $\mathsf{FL_e}$ have cut-free hypersequent calculi (Ciabattoni Galatos Terui 2008)

6. from above: lots of extensions of FLec and FLew have cut-free hypersequent calculi. Our results will apply to all these calculi

↪ representing hypersequents

Representing hypersequents in $(\mathcal{P}_f(\mathbb{N}^n))^{n+1}$

1. Let F_0 be empty formula
2. A hypersequent built from formulas $F_1, \ldots, F_n$ is written

$$\overbrace{X_1 \Rightarrow F_0}^{\text{sequent also called component}} \mid X_2 \Rightarrow F_0 \mid \ldots \mid X_{k_0} \Rightarrow F_0$$
$$Y_1 \Rightarrow F_1 \mid Y_2 \Rightarrow F_1 \mid \ldots \mid Y_{k_1} \Rightarrow F_1$$
$$\ldots$$
$$Z_1 \Rightarrow F_n \mid Z_2 \Rightarrow F_n \mid \ldots \mid Z_{k_n} \Rightarrow F_n$$

$\hookrightarrow$ hypersequent is an element of

Representing hypersequents in $(\mathcal{P}_f(\mathbb{N}^n))^{n+1}$

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$$\ldots$$
$$Z_1 \Rightarrow F_n \mid Z_2 \Rightarrow F_n \mid \ldots \mid Z_{k_n} \Rightarrow F_n$$

3. So a hypersequent is an element of

$$\underbrace{\mathcal{P}_f(\mathbb{N}^n) \times \mathcal{P}_f(\mathbb{N}^n) \times \ldots \times \mathcal{P}_f(\mathbb{N}^n)}_{n+1}$$

$\hookrightarrow$ what else to get FLec extensions

HFLec extensions: what do we need to extend?

1. absorb contraction by adding variant rules

$$\frac{h_1 \qquad\qquad h_N}{h_0}\ r \qquad\qquad \text{original}$$

$$\frac{h_1 \qquad\qquad h_N}{g}\ r^{(k,l)} \text{ with } h_0 \rightsquigarrow_c^k h' \rightsquigarrow_{EC}^l g \qquad \text{variants } k \le K, l \le L$$

2. For $(X_1, \ldots, X_d), (Y_1, \ldots, Y_d) \in (P_f(\mathbb{N}^n))^{n+1}$ define

   $(X_1, \ldots, X_{n+1}) \le_{\min} (Y_1, \ldots, Y_{n+1})$ iff $\forall y \in Y_i \exists x \in X_i (x \le y)$ for every i

3. $\mathbf{X} \le_{\min} \mathbf{Y}$ means we can go from $\mathbf{Y}$ to $\mathbf{X}$ by c, EC, EW

4. Using length function theorem for controlled bad sequences (Balasubramanian, 2020): decision problem for each of the FLec extensions under consideration is in $\mathbf{F}_{\omega^\omega}$

5. multiply-recursive functions composed with a single application of a hyper-Ackermannian function

$\hookrightarrow$ case of weakening

Extensions of HFL_{ew}: contraction replaced by weakening

$$\frac{X \Rightarrow A}{X, Y \Rightarrow A} \text{ weakening}$$

1. Prominent logic: monoidal t-norm based fuzzy logic

$$\mathsf{MTL} = \mathsf{FLew} + \mathsf{lin}$$

Describes the common behaviours of *all* fuzzy logics based on left-continuous t-norms

2. Previous argument insufficient when c replaced by w
If we encounter $(4,4)$ we can prohibit smaller elements like $(4,3)\ldots$

$$\frac{(4,3) \Rightarrow F}{(4,4) \Rightarrow F} \text{ height-preserving weakening}$$

But how to prohibit infinitely many larger elements? (infinite branch)

$$(4,4), (4,5), (4,6), \ldots, (4,100), \ldots$$

3. Time to go down the Lambek calculus [forward proof search]

↪ forward proof search

Forward proof search

Let S_0 be set of initial sequents built from subformulas in F

Def of S_1

$u \in S_0$

$\vdots$

limited weakening

$\vdots$

$$\frac{v}{w}\ r$$

$w \in S_1$

$u \in S_0$

$\vdots$

limited weakening

$\vdots$

surplus weakening

$$\frac{v}{w}\ r$$

$\rightsquigarrow$

$u \in S_0$

$\vdots$

limited weakening

$\vdots$

$$\frac{\exists v'}{w'}\ r$$

surplus weakening

w

1. Obtain $(S_0, S_1, \ldots)$ s.t. S_{i+1} finite and computable from S_i
2. what 'limited" means depends on the rules in the calculus

$\hookrightarrow$ what else to get FLew extensions

What do we need to extend?

1. a hypersequent is an element of $(\mathcal{P}_f(\mathbb{N}^n))^{n+1}$

2. For $(X_1, \ldots, X_d), (Y_1, \ldots, Y_d) \in (P_f(\mathbb{N}^n))^{n+1}$ define

   $(X_1, \ldots, X_{n+1}) \leq_{\text{maj}} (Y_1, \ldots, Y_{n+1})$ iff $\forall x \in X_i \exists y \in Y_i (x \leq y)$ for every i

3. $\mathbf{X} \leq_{\text{maj}} \mathbf{Y}$ means that we can go from $\mathbf{X}$ to $\mathbf{Y}$ by w, EC, EW

4. majoring ordering is a wqo so there exists N such that $S_{N+1} = S_N$

5. Using length function theorem (Balasubramanian 2020) to get max value for N: each FLew extensions under consideration is in $\mathbf{F}_{\omega^\omega}$

$\hookrightarrow$ no weakening and no contraction

Uninorm Fuzzy Logic: HFLe + com

Is HFLe + com decidable ?

1. We have seen that HFLec + com and HFL_{ew} + com are in $\mathbf{F}_{\omega^\omega}$
2. Every proof in HFLe + com is a proof in HFLec + com/HFL_{ew} + com
3. (don't use the variant rules, just the original rules)
4. We saw that HFLec + com has a finite proof search tree for F
5. ?? so proof search tree in HFLe + com for F must be a subtree ??

$\hookrightarrow$ is it?

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Is $\text{HFLe} + \text{com}$ decidable ?

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3. (don't use the variant rules, just the original rules)
4. We saw that $\text{HFLec} + \text{com}$ has a finite proof search tree for F
5. trap we truncated the proof search tree because we had contraction - without contraction this truncation is no longer justified
6. Decidability of uninorm logic UL is open

$\hookrightarrow$ further qs

Further questions and conversation starters

1. Can we find a logic in $\mathbf{F}_{\omega^\omega} - \mathbf{F}_\omega$

$$\dfrac{}{\text{cut-freeness seems to need } \underline{\text{hypersequents seem to yield } \mathbf{F}_{\omega^\omega}}}$$

2. lower bound and tighter bounds for MTL

   This was first syntactic proof and first complexity bound for MTL (many would suspect that more modest bounds should hold)

3. Simpler problem: lower bounds for FLec / FLew +

$$\frac{X, X, Z \Rightarrow F \qquad Y, Y, Z \Rightarrow F}{X, Y, Z \Rightarrow F}\ \text{scom}$$

'double antecedent, share between premises'

For example

$$\frac{p, q^4 \Rightarrow \qquad p^3, q^2 \Rightarrow}{p^2, q^3 \Rightarrow}$$

What type of (counter?) machine does this resemble?

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