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Herbrand Complexity and the Epsilon Calculus

(the case with equality)

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Herbrand Complexity

The optimal calculation of Herbrand disjunctions from unformalized or formalized mathematical proofs is one of the most prominent problems in proof theory of first-order logic.[1]

[1] A Sequent-Calculus Based Formulation of the Extended First Epsilon Theorem. Baaz et al. LFCS 2018

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Theorem

if $\exists \vec{x} E(\vec{x})$ *is a purely existential formula containing only the bound variables* $\vec{x}$, *and* $\mathrm{PC} \vdash_{\pi} \exists \vec{x} E(\vec{x})$ *then there are terms* t^i_j *such that*

$$\bigvee_{i=1}^{n} E(t^i_1, \ldots, t^i_m) \qquad \textcolor{red}{\textit{Herbrand disjunction}}$$

is a tautology

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is a tautology

the Herbrand complexity is the length n of the shortest Herbrand disjunction

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  2. equality axioms
  3. $A(t) \rightarrow A(\varepsilon_x A(x))$ critical axioms

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an ε-proof is a tautology $(\bigwedge_{i=1}^{n} A_i(t_i) \to A(\varepsilon_x A_i(x))) \to [\exists\, \vec{x} E(\vec{x})]^{\varepsilon}$

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  (someI in Isabelle/HOL)
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Outline

- **Why Bother?**
- **Epsilon Calculus with Equality**
- **Epsilon Calculus with Epsilon Equality**

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Why Bother?

Where Does the Epsilon Calculus Come From?

Rough Timeline

1922 introduced by Hilbert in 1921, as the basis for a formulation of mathematics for which his program was supposed to be carried out

1930s original work in proof theory (pre-Gentzen) concentrated on the ε-calculus and ε-substitution method (Ackermann, von Neumann, Bernays)

1950s ε-substitution method used by Kreisel for no-counterexample interpretation leading to work on proof analysis by Kreisel, Luckhardt, Kohlenbach

1990s use of the ε-substitution method for ordinal analysis by Arai, Avigad, Mints, Tait

recent renewed interest in connection to structural proof theory, update procedures and learning: Avigad, Aschieri, Baaz, Leitsch, Lolic, Powell . . .

Why Have You Never Heard of It, Though?

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Example

consider the embedding of $\exists x \exists y \exists z\, A(x, y, z)$, which yields

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$$
\begin{gathered}
A(\varepsilon_x A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), \varepsilon_z A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), z)),\\
\varepsilon_y A(\varepsilon_x A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), \varepsilon_z A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), z)), y,\\
\varepsilon_z A(\varepsilon_x A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), \varepsilon_z A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), z)), y, z)),\\
\varepsilon_z A(\varepsilon_x A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), \varepsilon_z A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)),\\
\varepsilon_y A(\varepsilon_x A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), \varepsilon_z A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), y,\\
\varepsilon_z A(\varepsilon_x A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), \varepsilon_z A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), z))))))))))
\end{gathered}
$$

Why Have You Never Heard of It, Though?

```
28:                                               ; preds = %25, %0
%29 = load i32, i32* %2, align 4
%30 = trunc i32 %29 to i8
%31 = load i8*, i8** %7, align 8
store i8 %30, i8* %31, align 1
%32 = load i8*, i8** %6, align 8
%33 = load i64, i64* %4, align 8
%34 = getelementptr inbounds i8, i8* %32, i64 %33
%35 = getelementptr inbounds i8, i8* %34, i64 -1
%36 = load i8*, i8** %7, align 8
%37 = icmp ule i8* %35, %36
br i1 %37, label %38, label %121
```

LLVM bytecode

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\end{gathered}$$

perhaps an ε-proof should be conceived as an internal representation, rather than something one writes down explicitely

Axioms of the Epsilon Calculus

Definitions

- AxEC: all propositional tautologies $+$ substitution instances of equality axioms:

$$s = s \qquad s = t \to f(\vec{u}, s, \vec{v}) = f(\vec{u}, t, \vec{v}) \qquad s = t \to (P(\vec{u}, s, \vec{v}) \to P(\vec{u}, t, \vec{v}))$$

- AxEC$_\varepsilon$: AxEC + all substitution instances of

$$A(t) \to A(\varepsilon_x A(x)) \qquad \text{(critical axiom)}$$

- AxEC$_\varepsilon^=$: AxEC$_\varepsilon$ + all substitution instances of

$$s = t \to \varepsilon_x A(x, \vec{u}, s, \vec{v}) = \varepsilon_x A(x, \vec{u}, t, \vec{v}) \qquad (\varepsilon\text{-equality axiom})$$

- AxPC: AxEC + all substitution instances of

$$A(a) \to \exists x\, A(x) \qquad \forall x\, A(x) \to A(a)$$

- AxPC$_\varepsilon^{(=)}$: AxPC + all substitution instances of critical formulas (and ε-equality ax.)

Definitions

- a proof in EC ($\mathrm{EC}_{\varepsilon}^{=}$) is a sequence $A_1, \ldots, A_n$ of formulas such that each A_i is either in AxEC ($\mathrm{AxEC}_{\varepsilon}^{=}$) or it follows from formulas preceding it by modus ponens
- a proof in PC ($\mathrm{PC}_{\varepsilon}^{=}$) is a sequence $A_1, \ldots, A_n$ of formulas such that each A_i is either in AxPC ($\mathrm{AxPC}_{\varepsilon}^{=}$) or follows from formulas preceding it by modus ponens or generalisation
- if A is provable in say $\mathrm{EC}_{\varepsilon}$ we write $\mathrm{EC}_{\varepsilon} \vdash_{\pi} A$

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- the size $\mathrm{sz}(\pi)$ of a proof π is the number of steps in π

- the critical count $\mathrm{cc}(\pi)$ of π is the number of distinct critical formulas, ε-equality axioms and quantifier axioms in π (plus 1)

quantifiers in a quantifier-free system:

$$\exists x\, A(x) \Leftrightarrow A(\varepsilon_x A(x)) \qquad \forall x\, A(x) \Leftrightarrow A(\varepsilon_x \neg A(x))$$

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define a mapping $^{\varepsilon}$:

$$f(t_1, \ldots, t_n)^{\varepsilon} = f(t_1^{\varepsilon}, \ldots, t_n^{\varepsilon})$$

$$x^{\varepsilon} = x \qquad [\varepsilon_x A(x)]^{\varepsilon} = \varepsilon_x A^{\varepsilon}(x)$$

$$a^{\varepsilon} = a$$

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Embedding Lemma

if π is a PC-proof of A then there is an EC_ε-proof π^ε of A^ε with $\mathrm{cc}(\pi^\varepsilon) \leqslant \mathrm{cc}(\pi)$

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if π is a regular PC-proof of A then there is an $\mathrm{EC}_{\varepsilon}$-proof π^{ε} of A^{ε} with $\mathrm{cc}(\pi^{\varepsilon}) \leqslant \mathrm{cc}(\pi)$

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Embedding Lemma (with equality)

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Example: Epsilon Mapping

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$$[\exists x(P(x) \quad \vee \quad \forall y Q(y))]^{\varepsilon} =$$

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\begin{aligned}
[\exists x(P(x) \quad &\vee \quad \forall y Q(y))]^{\varepsilon} = \\
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\end{aligned}
$$

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\end{aligned}
$$

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\end{aligned}
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= \quad & P(\underbrace{\varepsilon_x [P(x) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})]}_{e_2}) \vee Q(\underbrace{\varepsilon_y \neg Q(y)}_{e_1})
\end{aligned}
$$

Example (Drinker's Paradox)

$$
\begin{array}{c}
P(a) \Rightarrow P(a) \\
\hline
P(a) \Rightarrow P(a), \forall y P(y) \\
\hline
\Rightarrow P(a) \to \forall y P(y), P(a) \\
\hline
\Rightarrow \exists x (P(x) \to \forall y P(y)), P(a) \\
\hline
\Rightarrow \exists x (P(x) \to \forall y P(y)), \forall y P(y) \\
\hline
P(b) \Rightarrow \exists x (P(x) \to \forall y P(y)), \forall y P(y) \\
\hline
\Rightarrow \exists x (P(x) \to \forall y P(y)), P(b) \to \forall y P(y) \\
\hline
\Rightarrow \exists x (P(x) \to \forall y P(y)), \exists x (P(x) \to \forall y P(y)) \\
\hline
\Rightarrow \exists x (P(x) \to \forall y P(y))
\end{array}
$$

Example (Drinker's Paradox)

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P(a) \Rightarrow P(a)}{P(a) \Rightarrow P(a), \forall y P(y)}}{\Rightarrow P(a) \to \forall y P(y), P(a)}}{\Rightarrow \exists x(P(x) \to \forall y P(y)), P(a)}}{\Rightarrow \exists x(P(x) \to \forall y P(y)), \forall y P(y)}}{P(b) \Rightarrow \exists x(P(x) \to \forall y P(y)), \forall y P(y)}}{\Rightarrow \exists x(P(x) \to \forall y P(y)), P(b) \to \forall y P(y)}}{\Rightarrow \exists x(P(x) \to \forall y P(y)), \exists x(P(x) \to \forall y P(y))}}{\Rightarrow P(\varepsilon) \to P(\varepsilon_y \neg P(y))}$$

where we employ

$$[\forall y P(y)]^{\varepsilon} = P(\varepsilon_y \neg P(y))$$

$$[\exists x(P(x) \to \forall y P(y)]^{\varepsilon} = P(\underbrace{\varepsilon_x(P(x) \to P(\varepsilon_y \neg P(y)))}_{\varepsilon}) \to P(\varepsilon_y \neg P(y))$$

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\hline
\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(a) \\
\hline
\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), \forall y P(y) \\
\hline
P(b) \Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), \forall y P(y) \\
\hline
\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(b) \rightarrow \forall y P(y) \\
\hline
\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)) \\
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\end{array}
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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Rightarrow P(a)}{P(a) \Rightarrow P(a), P(\varepsilon_y \neg P(y))}}{\Rightarrow P(a) \rightarrow P(\varepsilon_y \neg P(y)), P(a)}}{\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(a)}}{\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y))}}{P(b) \Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(b) \rightarrow P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y))}}{\Rightarrow P(\varepsilon) \rightarrow P(\varepsilon_y \neg P(y))}
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$$
\begin{aligned}
[\forall y P(y)]^{\varepsilon} &= P(\varepsilon_y \neg P(y)) \\
[\exists x (P(x) \rightarrow \forall y P(y)]^{\varepsilon} &= P(\underbrace{\varepsilon_x (P(x) \rightarrow P(\varepsilon_y \neg P(y))))}_{\varepsilon}) \rightarrow P(\varepsilon_y \neg P(y))
\end{aligned}
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Example (Drinker's Paradox)

$$
\begin{array}{c}
P(\varepsilon_y \neg P(y)) \Rightarrow P(\varepsilon_y \neg P(y)) \\
\hline
P(\varepsilon_y \neg P(y)) \Rightarrow P(\varepsilon_y \neg P(y)), P(\varepsilon_y \neg P(y)) \\
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where we employ

$$[\forall y P(y)]^{\varepsilon} = P(\varepsilon_y \neg P(y))$$

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Drinker's Paradox (cont'd)

Example (cont'd)

1	$P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y))$	TAUT
2	$(P(\varepsilon_y \neg P(y)) \to P(\varepsilon_y \neg P(y))) \to$ $\to (P(\varepsilon_x (P(x) \to P(\varepsilon_y \neg P(y))))) \to P(\varepsilon_y \neg P(y)))$	critical axiom
3	$P(\varepsilon_x (P(x) \to P(\varepsilon_y \neg P(y))))) \to P(\varepsilon_y \neg P(y))$	$1, 2, MP$

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Remarks

- ε-calculus allows proof compression, eg. due to quantifier-shifts
- propositional inferences and structural rules become irrelevant
- focus on quantifier inferences

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- ε-calculus allows proof compression, eg. due to quantifier-shifts (see eg. Aguilera, Baaz, *Unsound Inferences Make Proofs Shorter*, JSL 2019)
- propositional inferences and structural rules become irrelevant
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Epsilon Calculus with Equality

The Extended First Epsilon Theorem

(w/o ε-Equality Axioms)

Theorem

Suppose $E(a_1, \dots, a_m)$ is quantifier-free and $s_1, \dots, s_m$ are ε-terms such that

$$\mathrm{EC}_\varepsilon \vdash_\pi E(s_1, \dots, s_m)$$

Then there are ε-free terms t^i_j such that

$$\mathrm{EC} \vdash \bigvee_{i=1}^{n} E(t^i_1, \dots, t^i_m)$$

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number of instances independent of # of propositional inferences

Herbrand's Theorem

(w/o ε-Equality Axioms)

Theorem

If $\exists x_1 \ldots \exists x_m E(x_1, \ldots, x_m)$ is a purely existential formula containing only the bound variables $x_1, \ldots, x_m$, and

$$\mathrm{PC} \vdash_\pi \exists x_1 \ldots \exists x_m E(x_1, \ldots, x_m)$$

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upper bound on Herbrand complexity independent of # of propositional inferences

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upper bound on Herbrand complexity independent of # of propositional inferences

Term Complexity of Herbrand Disjunction

Corollary

If $\exists\vec{x}E(\vec{x})$ is a purely existential formula containing only the bound variables $x_1, \ldots, x_m$, and

$$\mathrm{PC} \vdash_\pi \exists x_1 \ldots \exists x_m E(x_1, \ldots, x_m)\ ,$$

then there exists a primitive recursive function g and ε-free terms t^i_j such that

$$\mathrm{EC} \vdash \bigvee_{i=1}^{n} E(t^i_1, \ldots, t^i_m)$$

where $n, \mathrm{dp}(t^i_j) \leqslant g(\mathrm{cc}(\pi), \mathrm{ld}(E(\vec{x})))$

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Proof.

employ unification ∎

Proof of the First Epsilon Theorem (w/o $=$)

Simplifications

- suppose $\mathrm{EC}_\varepsilon \vdash_\pi E$ and E contains no ε-terms
- we show that $\mathrm{EC} \vdash E$ by induction on a term measure of π
- w.l.o.g. π doesn't contain any free variables (replace free variables by new constants—may be resubstituted later)

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- w.l.o.g. π doesn't contain any free variables (replace free variables by new constants—may be resubstituted later)

Lemma

Let $\pi \vdash A$, let e be a critical ε-term in π of "maximal" term measure. Then $\pi_e \vdash A$, where π_e is of "smaller" term measure

Proof of Lemma.

Construct π_e as follows:

1. Suppose $A(t_1) \to A(e), \ldots, A(t_n) \to A(e)$ are all the critical formulas belonging to e. For each critical axiom ($i = 1, \ldots, n$)

$$A(t_i) \to A(e)$$

we obtain a derivation

$$\pi_i \vdash A(t_i) \to E$$

as follows:

Proof of Lemma.

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   - Add $A(t_i)$ to the axioms. Now every such formula is derivable using the propositional tautology $A(t_i) \to (B \to A(t_i))$ and modus ponens

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- Apply the deduction theorem for the propositional calculus to obtain π_i

Proof (cont'd).

2. Obtain a derivation π' of

$$\bigwedge \neg A(t_i) \to E$$

by:

Proof (cont'd).

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3. Combine the proofs

$$\pi_i \vdash A(t_i) \to E$$

and

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Proof of the Extended First Epsilon Theorem

Theorem (Extended First Epsilon Theorem)

Suppose $E(a_1, \dots, a_m)$ *is quantifier-free and* $s_1, \dots, s_m$ *are* ε*-terms such that* $\mathrm{EC}_\varepsilon \vdash_\pi E(s_1, \dots, s_m)$*. Then there are* ε*-free terms* t^i_j *such that*

$$\mathrm{EC} \vdash \bigvee_{i=1}^{n} E(t^i_1, \dots, t^i_m)$$

where $n \leqslant 2^{3 \cdot \mathrm{cc}(\pi)}_{2 \cdot \mathrm{cc}(\pi)}$

Proof.

- suppose now the endformula E does contain ε-term
- ε-elimination method produces Herbrand disjunction of E by construction

∎

Theorem (Herbrand's Theorem)

If $\exists x_1 \ldots \exists x_k E(x_1, \ldots, x_k)$ is a purely existential formula containing only the bound variables $x_1, \ldots, x_k$, and PC $\vdash \exists x_1 \ldots \exists x_k E(x_1, \ldots, x_k)$. Then there are terms t_{ij} such that

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Proof.

- consider PC $\vdash_\pi \exists x_1 \ldots \exists x_k E(x_1, \ldots, x_k)$
- employing embedding we obtain $\mathrm{EC}_\varepsilon \vdash E(s_1, \ldots, s_k)$, where $s_1, \ldots, s_k$ are terms (containing ε's)
- employ the Extended First Epsilon Theorem

∎

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Epsilon Calculus with Epsilon Equality

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Definition ("Grundtyp")

An ε-term $\varepsilon_x A(x)$ is an ε-matrix if the only terms that occur in $\varepsilon_x A(x)$ are free variables, each of which occurs exactly once.

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Definition (revisited)

$\mathrm{AxEC}_{\varepsilon}^{=}$: AxEC + all substitution instances of critical formulas + all substitution instances of

$$s = t \to \varepsilon_x A(x, \vec{u}, s, \vec{v}) = \varepsilon_x A(x, \vec{u}, t, \vec{v})$$

where $\varepsilon_x A(x, \vec{b}, a, \vec{c})$ is an ε-matrix

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Remark

the restriction to ε-matrices for ε-equality axioms is crucial

Digression: Unrestricted Epsilon Equality Axioms

let $\mathrm{EC}'^{=}_{\varepsilon}$ denote the extension of the ε-calculus $\mathrm{EC}_{\varepsilon}$ with the following axioms to cover ε-equality:

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Theorem

There exists an existential formulas $\exists \vec{x} E(\vec{x})$ such that

$$\mathrm{EC}'^{=}_{\varepsilon} \vdash_{\pi} [\exists \vec{x} E(\vec{x})]^{\varepsilon}$$

but we cannot show existence of ε-free terms $\vec{t}_0, \vec{t}_1, \ldots, \vec{t}_n$ such that $\mathrm{EC} \vdash \bigvee_{i=0}^{n} E(\vec{t}_i)$ and n is bounded in the size π

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Proof.

apply Yukami's trick ∎

The Extended First Epsilon Theorem

(with ε-Equality Axioms)

Theorem (First Epsilon Theorem with Epsilon Equality)

Suppose $E(a_1, \ldots, a_m)$ is quantifier-free and $s_1, \ldots, s_m$ are ε-terms such that

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where $n \leqslant 2^{\mathrm{cc}(\pi)+\mathsf{mpd}(\pi)}_{3 \cdot \mathrm{cc}(\pi)+2}$

Proof Sketch.

- remove critical axioms belonging to "maximal" ε-terms
- let the following ε-equality axioms belong to ε-term e

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- this step may require to "repair" critical axioms by derived identity schemas:

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Term Complexities Revisited

Corollary

Suppose $E(a_1, \ldots, a_m)$ *is a quantifier-free and* $s_1, \ldots, s_m$ *are* ε*-terms, such that* $\mathrm{EC}_\varepsilon^= \vdash_\pi E(s_1, \ldots, s_m)$. *Then there exists a primitive recursive function* g *and* ε*-free terms* t_j^i *such that*

$$\mathrm{EC} \vdash \bigvee_{i=1}^{n} E(t_1^i, \ldots, t_m^i)$$

where $n, \mathrm{dp}(t_j^i) \leqslant g(\mathrm{cc}(\pi), \mathrm{mpd}(\pi), \mathrm{ld}(E(\vec{x})))$

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Remark

- the presence of ε-equality axioms makes the ε-elimination (much) more involved
- again a bound on the Herbrand complexity can be read off, however depending not only on the $\mathrm{cc}(\pi)$, but also on properties ($\mathrm{mpd}(\pi)$) of ε-matrices in π

Conclusion and Open Questions

Final Remarks

1. two results on Herbrand complexity
   - Herbrand complexity depends on the critical count of the initial proof (w/o ε-equality formulas)
   - Herbrand complexity depends on the critical count of the initial proof and term complexity of ε-equality formulas
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Open Questions

1. significant gap between lower/upper bound
2. sequent calculus representation, like the Mints-Yasuhara system, that admits syntactic cut-elimination

Thank You for Your Attention!