Focusing Gentzen's LK proof system

Dale Miller

Inria Saclay & LIX, École Polytechnique
Palaiseau, France

The Proof Theory Virtual Seminar, 3 March 2021

This talk is based on the paper [Liang and Miller, 2020], co-authored with Chuck Liang (available from my web page).

What is a good foundation for Computational Logic?

Computational logic covers many topics.

- Theorem proving: classical, intuitionistic, inductive
- Model checking
- Logic programming
- Type checking and inference
- Curry-Howard Correspondence
- SAT, SMTP, etc.

Claim: Proof theory can provide a strong foundation for all these topics, but Gentzen's original systems need to be updated.

This talk focuses on such an update for Gentzen's LK.

Gentzen's LK using two-sided sequents

Structural rules

$$\frac{\Gamma, B, B \vdash \Delta}{\Gamma, B \vdash \Delta}\ cL \qquad \frac{\Gamma \vdash \Delta, B, B}{\Gamma \vdash \Delta, B}\ cR \qquad \frac{\Gamma \vdash \Delta}{\Gamma, B \vdash \Delta}\ wL \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, B}\ wR$$

Identity rules

$$\frac{}{B \vdash B}\ init \qquad \frac{\Gamma \vdash \Delta, B \quad \Gamma, B \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut$$

Introduction rules

$$\frac{\Gamma, B_i \vdash \Delta}{\Gamma, B_1 \wedge B_2 \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, B \quad \Gamma \vdash \Delta, C}{\Gamma \vdash \Delta, B \wedge C} \qquad \frac{}{\Gamma \vdash \Delta, t}$$

$$\frac{\Gamma, B \vdash \Delta \quad \Gamma, C \vdash \Delta}{\Gamma, B \vee C \vdash \Delta} \qquad \frac{}{\Gamma, f \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, B_i}{\Gamma \vdash \Delta, B_1 \vee B_2}$$

$$\frac{\Gamma \vdash \Delta, B \quad \Gamma, C \vdash \Delta'}{\Gamma, \Gamma', B \supset C \vdash \Delta, \Delta'} \qquad \frac{\Gamma, B \vdash \Delta, C}{\Gamma \vdash \Delta, B \supset C}$$

$$\frac{\Gamma, Bs \vdash \Delta}{\Gamma, \forall x.Bx \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, By}{\Gamma \vdash \Delta, \forall x.Bx} \qquad \frac{\Gamma, By \vdash \Delta}{\Gamma, \exists x.Bx \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, Bs}{\Gamma \vdash \Delta, \exists x.Bx}$$

Observations about LK

The structural rule of exchange is built into this presentation.

The additive variants of conjunction and disjunction are used, not the multiplicative variants.

Implication is multiplicative (hence, a kind of multiplicative disjunction).

Gentzen used negation $\neg B$ while here we use $B \supset f$. As a result, the LJ restriction on LK can be stated as either

- there is exactly one formula on the right, or
- the right is a *linear* context, the left is a *classical* context.

Thus, intuitionistic logic is a hybridization of linear and classical logics. We shall seldom mention intuitionistic logic in this talk.

Four shortcomings of the LK sequent calculus

1. The collision of cut and the structural rules
2. Permutations of inference rules
3. Chose either the additive or multiplicative versions of binary inference rules, but not both
4. No provision for synthetic inference rules

1: The collision of cut and the structural rules

Consider the following instance of the cut rule.

$$\frac{\Gamma \vdash C \qquad \Gamma', C \vdash B}{\Gamma, \Gamma' \vdash B}\ \mathit{cut}$$

1: The collision of cut and the structural rules

Consider the following instance of the cut rule.

$$\dfrac{\Gamma \vdash C \qquad \dfrac{\Gamma', C, C \vdash B}{\Gamma', C \vdash B}}{\Gamma, \Gamma' \vdash B}\ cut$$

If the right premise is proved by a left-contraction rule from the sequent $\Gamma', C, C \vdash B$, then permute the *cut* rule to the right:

$$\dfrac{\dfrac{\Gamma \vdash C \qquad \dfrac{\Gamma \vdash C \qquad \Gamma', C, C \vdash B}{\Gamma, \Gamma', C \vdash B}\ cut}{\Gamma, \Gamma, \Gamma' \vdash B}\ cut}{\Gamma, \Gamma' \vdash B}\ cL.$$

1: The collision of cut and the structural rules

Consider the following instance of the cut rule.

$$\dfrac{\dfrac{\Gamma \vdash C, C}{\Gamma \vdash C} \qquad \Gamma', C \vdash B}{\Gamma, \Gamma' \vdash B}\ cut$$

If the left premise is proved by a right-contraction rule from the sequent $\Gamma \vdash C, C$, then permute the *cut* rule to the left:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash C, C \quad \Gamma', C \vdash B}{\Gamma, \Gamma' \vdash C, B}\ cut \qquad \Gamma', C \vdash B}{\Gamma, \Gamma', \Gamma' \vdash B, B}\ cut}{\Gamma, \Gamma' \vdash B}\ cL,\ cR$$

1: The collision of cut and the structural rules

Consider the following instance of the cut rule.

$$\dfrac{\dfrac{\Gamma \vdash C, C}{\Gamma \vdash C} \qquad \dfrac{\Gamma', C, C \vdash B}{\Gamma', C \vdash B}}{\Gamma, \Gamma' \vdash B}\ cut$$

What if both premises are contractions? Cut can *non-deterministically* move to either premises.

In intuitionistic logic, this non-determinism is avoided since contraction on the right is simply forbidden.

1: The collision of cut and the structural rules (continued)

Such nondeterminism in cut-elimination is even more pronounced when we consider the collision of the cut rule with weakening.

$$\dfrac{\dfrac{\dfrac{\Xi_1}{\vdash B}}{\vdash C, B}\ wR \qquad \dfrac{\dfrac{\Xi_2}{\vdash B}}{C \vdash B}\ wL}{\dfrac{\vdash B, B}{\vdash B}\ cR}\ cut$$

Cut-elimination can yield either Ξ_1 or Ξ_2.

This kind of example does not occur in the intuitionistic (single-sided) version of the sequent calculus.

These are often called *Lafont's examples* [Girard et al., 1989].

Polarization will allow us to say something more general.

2. Permutations of inference rules

The following two deviations differ by permuting an inference rule.

$$\dfrac{\dfrac{\Gamma, B_i, C_j \vdash \Delta}{\Gamma, B_i, C_1 \wedge C_2 \vdash \Delta}}{\Gamma, B_1 \wedge B_2, C_1 \wedge C_2 \vdash \Delta} \qquad \dfrac{\dfrac{\Gamma, B_i, C_j \vdash \Delta}{\Gamma, B_1 \wedge B_2, C_j \vdash \Delta}}{\Gamma, B_1 \wedge B_2, C_1 \wedge C_2 \vdash \Delta}$$

These two derivations are different but should be considered equal.

Permutation of inference rules is a huge issue in trying to see structure in the sequent calculus.

The existence of such permutations is probably the main reason for the revolt again sequent calculus, giving rise to natural deduction/typed λ-calculi, expansion trees, proof nets, etc.

3. Choose only one among additive or multiplicative rules

Gentzen used the additive versions of conjunction and disjunction.

People in theorem proving usually use the invertible rules for conjunction and disjunction (which is multiplicative). Things can then be arranged so that the only non-invertible rule is the $\exists$R rule.

Why not allow *both* the additive and multiplicative versions of these rules?

4. No provision for synthetic inference rules

Inference rules in LK are too small. Consider the axiom stating that the predicate *path* is transitive.

$$\forall x \forall y \forall z\,(\mathit{path}\ x\ y \supset \mathit{path}\ y\ z \supset \mathit{path}\ x\ z).$$

Using this axiom involves at least five LK introduction rules. It is more natural to view that formula as yielding an inference rule.

$$\frac{\Gamma \vdash \Delta,\ \mathit{path}\ x\ y \quad \Gamma \vdash \Delta,\ \mathit{path}\ y\ z}{\Gamma \vdash \Delta, \mathit{path}\ x\ z}$$

$$\frac{\mathit{path}\ x\ y,\ \mathit{path}\ y\ z,\ \mathit{path}\ x\ z,\ \Gamma \vdash \Delta}{\mathit{path}\ x\ y,\ \mathit{path}\ y\ z, \Gamma \vdash \Delta}$$

One of these *synthetic rules* would be a more appropriate way to invoke the transitivity axiom.

How can we build such synthetic rules? Can we guarantee cut-elimination holds when we add them?

LKF: polarized formulas

Positive connectives are f^+, $\vee^+$, t^+, $\wedge^+$, and $\exists$.
Negative connectives are t^-, $\wedge^-$, f^-, $\vee^-$, and $\forall$.
Literals are atomic formulas and negated atomic formulas.

An *atomic bias assignment* is a function $\delta(\cdot)$ that maps atomic formulas to the set $\{+, -\}$.

We extended $\delta(\cdot)$ to literals by setting $\delta(\neg A)$ to the opposite polarity of $\delta(A)$.

A polarized formula is *positive* if its top-level connective is positive or its a literal L and $\delta(L) = +$.
A polarized formula is *negative* if its top-level connective is negative or its a literal L and $\delta(L) = -$.

We require that $\delta(\cdot)$ is stable under substitution: $\delta(A) = \delta(\theta A)$.
Thus, $\delta(A)$ is determined by the predicate symbol of A.

LKF: polarized formulas (continued)

Linear logic has other names for the polarized connectives.

	conjunction	true	disjunction	false
multiplicative	$\wedge^+$, $\otimes$	t^+, 1	$\vee^-$, $\parr$	f^-, $\bot$
additive	$\wedge^-$, $\&$	t^-, $\top$	$\vee^+$, $\oplus$	f^+, 0

Logical connectives have *four attributes:*
arity, additive/multiplicative, polarity, conjunction/disjunction.

De Morgan duality flips the last two but leaves the first two unchanged.

LKF: negation normal form

Polarized formulas are in *negation normal form* (nnf), meaning that there is no occurrences of implication $\supset$ and that the negation symbol $\neg$ has only atomic scope.

The negation symbol $\neg$ is extended also to non-atomic polarized formulas.

- $\neg\neg A = A$ for atomic formula A
- $\neg(A \wedge^+ B) = \neg A \vee^- \neg B, \quad \neg(A \vee^- B) = \neg A \wedge^+ \neg B$
- $\neg(A \vee^+ B) = \neg A \wedge^- \neg B, \quad \neg(A \wedge^- B) = \neg A \vee^+ \neg B$
- $\neg\exists x.A = \forall x.\neg A, \quad \neg\forall x.A = \exists x.\neg A$

Let B be an unpolarized formula (in nnf) and let $\hat{B}$ result from annotating the propositional connectives in B with a $+$ or $-$. Let $\delta(\cdot)$ be an atomic bias assignment for the predicates in B. The pair $\langle \delta(\cdot), \hat{B} \rangle$ is a *polarization* of B.

LKF: sequent

LKF uses one-sided sequents of two varieties, namely,

$$\vdash \Gamma \Uparrow \Theta \quad \text{and} \quad \vdash A \Downarrow \Theta,$$

where Γ is a multiset of formulas, Θ is a set of formulas, and A is a single formula.

I will call the Θ context *storage*.

Introductions take place on formulas between $\vdash$ and the $\Uparrow$ or $\Downarrow$.

LKF: proof rules

Negative introduction rules

$$\frac{}{\vdash t^-, \Gamma \Uparrow \Theta} \qquad \frac{\vdash A, \Gamma \Uparrow \Theta \qquad \vdash B, \Gamma \Uparrow \Theta}{\vdash A \wedge^- B, \Gamma \Uparrow \Theta}$$

$$\frac{\vdash \Gamma \Uparrow \Theta}{\vdash f^-, \Gamma \Uparrow \Theta} \qquad \frac{\vdash A, B, \Gamma \Uparrow \Theta}{\vdash A \vee^- B, \Gamma \Uparrow \Theta} \qquad \frac{\vdash [y/x]B, \Gamma \Uparrow \Theta}{\vdash \forall x.B, \Gamma \Uparrow \Theta}$$

Positive introduction rules

$$\frac{}{\vdash t^+ \Downarrow \Theta} \qquad \frac{\vdash A \Downarrow \Theta \qquad \vdash B \Downarrow \Theta}{\vdash A \wedge^+ B \Downarrow \Theta} \qquad \frac{\vdash B_i \Downarrow \Theta}{\vdash B_1 \vee^+ B_2 \Downarrow \Theta} \qquad \frac{\vdash [s/x]B \Downarrow \Theta}{\vdash \exists x.B \Downarrow \Theta}$$

Non-introduction rules

$$\frac{}{\vdash p \Downarrow \neg p, \Theta}\ \textit{init} \qquad \frac{\vdash \Gamma \Uparrow Q, \Theta}{\vdash Q, \Gamma \Uparrow \Theta}\ \textit{store} \qquad \frac{\vdash N \Uparrow \Theta}{\vdash N \Downarrow \Theta}\ \textit{release}$$

$$\frac{\vdash P \Downarrow P, \Theta}{\vdash \cdot \Uparrow P, \Theta}\ \textit{decide}$$

Here: p is a positive literal, P is positive, N is negative, Q is positive or a literal.

Observations about LKF proof rules

We say that the polarized formula B has an LKF proof if the sequent $\vdash B \Uparrow \cdot$ has an LKF proof

Storage (the Θ context) is non-decreasing as we move from conclusion to premise.

Key observations:

1. *Contraction* occurs only in the *decide* rule and only for *positive* formulas. A negative formula is never contracted.
2. *Weakening* occurs only at the leaves (in the *init* and t^+ rules) and only on *positive formulas* and *negative literals*.

Theorem (Completeness of LKF)

Let B be an unpolarized formula and a theorem of LK. If $\hat{B}$ is any polarization of B then $\hat{B}$ has an LKF proof.

[Liang and Miller, 2009] proves this using translations into intuitionistic and linear logics. [Liang and Miller, 2020] gives a direct proof.

The structure of (cut-free) focused proofs

A sequent of the form $\vdash \cdot \Uparrow \Theta$ is called a *border sequent*.

Such sequents can only be proved by using the *decide* rule.

A *synthetic inference rule* is defined as these two phases, with border sequents as the conclusion and the premises.

$$\dfrac{\dfrac{\cdots \qquad \dfrac{\cdots \qquad \dfrac{\dfrac{\cdots \quad \vdash \cdot \Uparrow \Theta' \quad \cdots}{\vdash N \Uparrow \cdots}\ \text{neg phase}}{\vdash N \Downarrow \cdots}\ \textit{release} \qquad \cdots}{\vdash \cdots \Downarrow \cdots}\ \text{pos phase} \qquad \cdots}{\vdash P \Downarrow \Theta}}{\vdash \cdot \Uparrow \Theta}\ \textit{decide}\ P \in \Theta$$

The cut rule for LKF

The *cut* rule operates on $\Uparrow$ sequents.

$$\frac{\vdash B \Uparrow \Theta \quad \vdash \neg B \Uparrow \Theta'}{\vdash \cdot \Uparrow \Theta, \Theta'}\ cut$$

During the proof of cut-elimination, the following four variants of the cut rule need to be considered and eliminated as well.

$$\frac{\vdash A, \Gamma \Uparrow \Theta \quad \vdash \neg A, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ cut_u \qquad \frac{\vdash A \Downarrow \Theta \quad \vdash \neg A, \Gamma' \Uparrow \Theta'}{\vdash \Gamma' \Uparrow \Theta, \Theta'}\ cut_f$$

$$\frac{\vdash \Gamma \Uparrow \Theta, P \quad \vdash \neg P, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ dcut_u \qquad \frac{\vdash B \Downarrow \Theta, P \quad \vdash \neg P \Uparrow \Theta'}{\vdash B \Downarrow \Theta, \Theta'}\ dcut_f$$

Here, A and B are arbitrary polarized formulas and P is a positive polarized formula.

Outline of completeness proof

1. Prove that all four cuts are admissible.
2. Prove the admissibility of the general *init* rule, sometimes called *init* expansion.
3. Prove some generalized invertibility lemmas.
4. Embed Gentzen's LK into LKF by choosing an appropriate polarization.
5. Prove that all LK rules are admissible in LKF.

Applications of LKF: Admissibility of *cut* in LK

Theorem

The cut rule for LK is admissible in the cut-free fragment of LK.

Follows immediately from the meta-theory of LKF.

Applications of LKF: Lafont's examples disappear

In all occurrences of the *cut* rule in LKF,

$$\frac{\vdash B \Uparrow \Theta \quad \vdash \neg B \Uparrow \Theta'}{\vdash \cdot \Uparrow \Theta, \Theta'}\ cut$$

exactly one of B and $\neg B$ is negative and one is positive. Hence, contraction is available only for one of these (the positive one) and not both.

Application of LKF: Synthetic inference rules

Let Θ contain the negated and polarized transitivity axiom:

$$\exists x \exists y \exists z.(path\ x\ y \wedge^+ path\ y\ z \wedge^+ \neg path\ x\ z)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Xi_1}{\vdash path\ r\ s \Downarrow \Theta} \quad \dfrac{\Xi_2}{\vdash path\ s\ t \Downarrow \Theta} \quad \dfrac{\Xi_3}{\vdash \neg path\ r\ t \Downarrow \Theta}}{\vdash path\ r\ s \wedge^+ path\ s\ t \wedge^+ \neg path\ r\ t \Downarrow \Theta}\ \wedge^+ \times 2}{\vdash \exists x \exists y \exists z.(path\ x\ y \wedge^+ path\ y\ z \wedge^+ \neg path\ x\ z) \Downarrow \Theta}\ \exists \times 3}{\vdash \cdot \Uparrow \Theta}\ decide$$

The shape of Ξ_1, Ξ_2, and Ξ_3 depends on the polarity of the *path* predicate.

Application of LKF: Synthetic inference rules (continued)

If path-atoms are negative, then Ξ_1 and Ξ_2 end with the *release* and *store* rules while the proof Ξ_3 is trivial. This synthetic rule is

$$\frac{\vdash \cdot \Uparrow \textit{path } r\ s, \Theta \qquad \vdash \cdot \Uparrow \textit{path } s\ t, \Theta}{\vdash \cdot \Uparrow \textit{path } r\ t, \Theta}$$

If path atoms are positive, then Ξ_3 end with the *release* and *store* rules while the proof Ξ_1 and Ξ_2 are trivial. This synthetic rule is

$$\frac{\vdash \cdot \Uparrow \neg\textit{path } r\ s, \neg\textit{path } s\ t, \neg\textit{path } r\ t, \Theta}{\vdash \cdot \Uparrow \neg\textit{path } r\ s, \neg\textit{path } s\ t, \Theta}$$

These synthetic inference rules are the one-sided version of the *back-chaining* and *forward-chaining* rules displayed earlier (see [Chaudhuri et al., 2008]).

Cut-elimination holds when synthetic inference rules are added [Marin et al., 2020].

Application of LKF: Herbrand's theorem

The formula $\exists\bar{x}.B$ is provable if and only if there are substitutions $\theta_1, \ldots, \theta_m$ ($m \geq 1$) such that $B\theta_1 \vee \cdots \vee B\theta_m$ is provable.

Let $\hat{B}$ be a polarized version of B in which all logical connectives in B are polarized negatively. Since $\exists\bar{x}.B$ is provable, the sequent $\vdash \exists\bar{x}.\hat{B} \Uparrow \cdot$ and $\vdash \cdot \Uparrow \exists\bar{x}.\hat{B}$ must have LKF proofs.

Let C be the formula $\hat{B}\theta_1 \vee^+ \ldots \vee^+ \hat{B}\theta_m$ where θ_i is the i^{th} instantiate of $\exists\bar{x}.B$ in that LKF proof.

$$\dfrac{\dfrac{\vdash \hat{B}\theta_i \Uparrow \exists\bar{x}.\hat{B}, \mathcal{L}}{\vdash \hat{B}\theta_i \Downarrow \exists\bar{x}.\hat{B}, \mathcal{L}}\ \textit{release}}{\vdash \exists\bar{x}.\hat{B} \Downarrow \exists\bar{x}.\hat{B}, \mathcal{L}}\ \exists \times n \qquad \Longrightarrow \qquad \dfrac{\dfrac{\vdash \hat{B}\theta_i \Uparrow C, \mathcal{L}}{\vdash \hat{B}\theta_i \Downarrow C, \mathcal{L}}\ \textit{release}}{\vdash \hat{B}\theta_1 \vee^+ \ldots \vee^+ \hat{B}\theta_m \Downarrow C, \mathcal{L}}\ \vee^+$$

Except for the details *inside* the $\Downarrow$-phase, these proofs are *identical*.

Application of LKF: Hosting other proof systems - Delays

Delays $\partial_-(\cdot)$ and $\partial_+(\cdot)$ can be inserted into formulas in order to break collections of connectives of the same polarity up into smaller blocks of connectives.

Delays can be defined in one of two ways.

- Define $\partial_-(B)$ as $f^- \vee^- B$, $t^- \wedge^- B$, or $\forall x B$ and define $\partial_+(B)$ as $f^+ \vee^+ B$, $t^+ \wedge^+ B$, or $\exists x B$ (vacuous binder in both cases).
- Define $\partial_-(B)$ as a 1-ary & or 1-ary $\parr$ and define $\partial_+(B)$ as a 1-ary $\oplus$ or 1-ary $\otimes$.

While B, $\partial_-(B)$, and $\partial_+(B)$ are logically equivalent, $\partial_-(B)$ is always negative and $\partial_+(B)$ is always positive.

Application of LKF: Hosting other proof systems

The LKQ and LKT proof systems of [Danos et al., 1995] can be seen as LKF proofs in which the following polarization functions are used. Here, A ranges over atomic formulas.

LKT	LKQ
Atoms are negative	Atoms are positive
$(A)^l = \neg A$	$(A)^l = \neg A$
$(A)^r = A$	$(A)^r = A$
$(B \supset C)^l = (B)^r \wedge^+ (C)^l$	$(B \supset C)^l = (B)^r \wedge^+ \partial_-((C)^l)$
$(B \supset C)^r = (B)^l \vee^- \partial_+((C)^r)$	$(B \supset C)^r = \partial_+((B)^l \vee^- (C)^r)$

Cut-free proofs in LKT (resp, LKQ) of B correspond to LKF proofs of $(B)^r$ using the LKT (resp, LKQ) definition.

Gentzen's LK proof system can also be hosted inside LKF by using lots of delays.

Variants for focusing in classical logic: multifocusing

Positive introduction rules

$$\frac{}{\vdash t^+ \Downarrow \Gamma} \qquad \frac{\vdash B_1, \Theta_1 \Downarrow \Gamma \quad \vdash B_2, \Theta_2 \Downarrow \Gamma}{\vdash B_1 \wedge^+ B_2, \Theta_1, \Theta_2 \Downarrow \Gamma} \qquad \frac{\vdash B_i, \Theta \Downarrow \Gamma}{\vdash B_1 \vee^+ B_2, \Theta \Downarrow \Gamma}\ i \in \{1,2\}$$

Release and decide rules

$$\frac{\vdash \Delta \Uparrow \Gamma}{\vdash \Delta \Downarrow \Gamma}\ \mathit{release}^\dagger \qquad \frac{\vdash \Delta \Downarrow \bar{\Delta}, \Gamma}{\vdash \cdot \Uparrow \bar{\Delta}, \Gamma}\ \mathit{decide}^\ddagger$$

The proviso $\dagger$: Δ consists of only negative formulas.
The proviso $\ddagger$: Δ is a non-empty multiset of positive formulas and $\bar{\Delta}$ is its underlying set.

Simple changes to LKF (to *init*, *decide*, and the introduction rules for $\otimes$ and 1) yields MALLF, a focused proof system for the MALL fragment of linear logic, first proposed in [Andreoli, 1992].

Conclusion

The LKF proof system is proposed as an improvement on LK, especially for computer scientists interested in computational logic.

The LKF proof system is flexible and can mimic a range of proof systems and supports the inclusion of synthetic inference rules.

The proof theory of LKF can be applied to unpolarized proof systems as well (e.g., Herbrand's theorem).

Intuitionistic logic can similarly be given a focused proof system LJF [Liang and Miller, 2009].

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