# TOTALLY LINEAR PROOFS FOR CLASSICAL LOGICS

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joint work with Chris Barrett and Victoria Barrett

Proof Society 15/12/21

Talk available from AG's home page and at https://people.bath.ac.uk/ag248/t/TLPCL.pdf

All about deep inference at http://alessio.guglielmi.name/res/cos

# EXAMPLE

$$(\bar{a} \wedge \bar{b}) \vee (a \vee b)$$

If $a = b = 0$ the conjunction is true, otherwise the disjunction is true.

# EXAMPLE

$$S_1 \equiv (\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)$$

$$\begin{aligned} S_2 \equiv \; & (\bar{a}_2 \wedge \bar{b}_2) \\ & \vee (((a_2 \vee b_2) \wedge \bar{a}_1) \wedge \\ & \quad ((a_2 \vee b_2) \wedge \bar{b}_1)) \\ & \vee (a_1 \vee b_1) \end{aligned}$$

If $a_2 = b_2 = 0$ the first conjunction is true, otherwise $S_2 = S_1$.

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$$S_1 \equiv (\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)$$

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If $a_2 = b_2 = 0$ the first conjunction is true, otherwise $S_2 = S_1$.

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If $a_3 = b_3 = 0$ the first conjunction is true, otherwise $S_3 = S_2$.

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If $a_3 = b_3 = 0$ the first conjunction is true, otherwise $S_3 = S_2$.

…

This case analysis is natural.

# EXAMPLE: STATMAN TAUTOLOGIES

**Definition** We call Statman tautologies the formulae $S_1, S_2, \ldots$:

$$S_1 \equiv (\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)$$

for $n > k \geqslant 1$

$$S_n \equiv$$

$$(\bar{a}_n \wedge \bar{b}_n) \vee ((A^n_{n-1} \wedge B^n_{n-1}) \vee \cdots \vee (A^n_1 \wedge B^n_1)) \vee (a_1 \vee b_1)$$

where:

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We work modulo associativity.

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…

In tree-like cut-free Gentzen systems, all proofs of Statman tautologies grow at least exponentially.

[Statman, Ann. Math. Logic, 1978]

Case analysis by cut yields polynomial proofs.

# EXAMPLE: STATMAN TAUTOLOGIES

**Claim** There exist cut-free proofs of Statman tautologies of size $O(n^{2.5})$ on the size $n$ of the tautologies.

$S_1 \equiv (\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)$ for $n > k \geqslant 1$

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**Idea** We build a cut-free derivation

$$\begin{array}{c} 1 \\ \Vert \\ S_1 \\ \Vert \\ \vdots \\ \Vert \\ S_n \end{array}$$

Each step must be a cut-free case analysis.

# EXAMPLE: STATMAN TAUTOLOGIES

**Claim** There exist cut-free proofs of Statman tautologies of size $O(m^{2.5})$ on the size $m$ of the tautologies.

$S_1 \equiv (\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)$ $O(m)$ for $n > k \geqslant 1$

$S_n \equiv$ $O(n^2)$

$(\bar{a}_n \wedge \bar{b}_n) \vee ((A^n_{n-1} \wedge B^n_{n-1}) \vee \cdots \vee (A^n_1 \wedge B^n_1)) \vee (a_1 \vee b_1)$

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**Idea** We build a cut-free derivation

$$O(m^{0.5}) \text{ steps} \left\{ \begin{array}{c} 1 \\ \| \\ S_1 \\ \| \\ \vdots \\ \| \\ S_n \end{array} \right.$$

Each step must be a cut-free case analysis.

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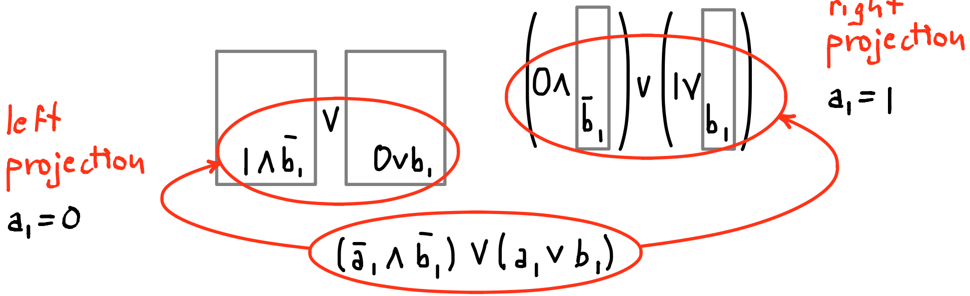
## Idea

Base case



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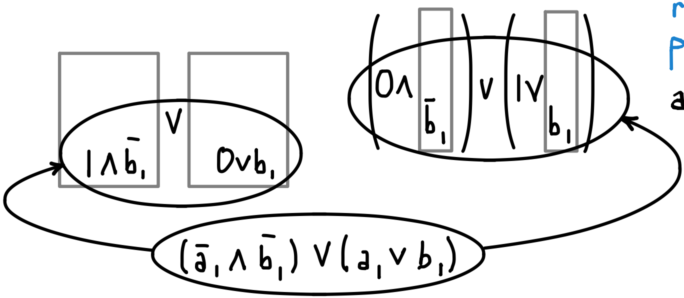
## Idea

## Base case



left projection $a_1 = 0$

right projection $a_1 = 1$

$a_i$ stands for $(0\,\mathbf{a}_i\,1)$ (whence 'subatomic')

$b_i$ stands for $(0\,\mathbf{b}_i\,1)$

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## Idea

Base case

decision tree constructor

proof are freely composed by connectives

$$\boxed{\boxed{1 \wedge \bar{b}_1} \vee \boxed{0 \vee b_1}} \; \textcircled{$a_1$} \; ((0 \wedge \boxed{\bar{b}_1}) \vee (1 \vee \boxed{b_1}))$$

$$(\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)$$

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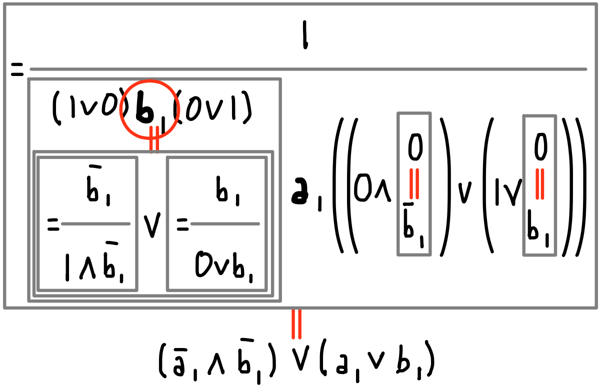
## Idea

### Base case

projection for $b_1$



$a_i$ stands for $(0\, \mathbf{a}_i\, 1)$ (whence 'subatomic')

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‖ are derivations to be specified (see later).

Natural case analysis.

# EXAMPLE: STATMAN TAUTOLOGIES

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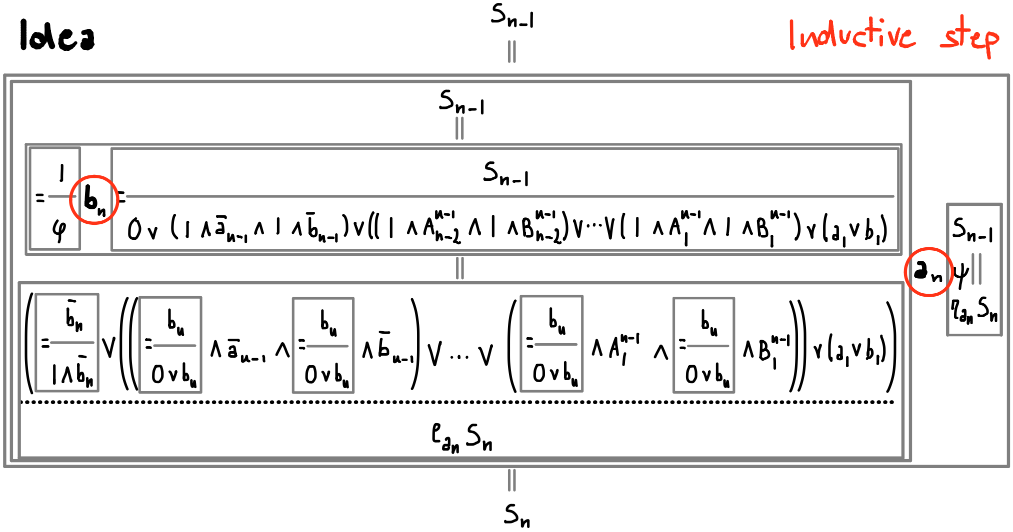
## Idea

Inductive step



Natural case analysis.

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**Claim** There exist cut-free proofs of Statman tautologies of size $O(n^{2.5})$ on the size m of the tautologies.

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## Idea

Inductive step

$$\begin{array}{c} S_{n-1} \\ \| \\ \boxed{\begin{array}{c} S_{n-1} \\ \| \\ \boxed{\boxed{=\dfrac{1}{\varphi}}\; b_n \;\boxed{=\dfrac{S_{n-1}}{0 \vee (1 \wedge \bar a_{n-1} \wedge 1 \wedge \bar b_{n-1}) \vee ((1 \wedge A^{n-1}_{n-2} \wedge 1 \wedge B^{n-1}_{n-2}) \vee \cdots \vee (1 \wedge A^{n-1}_1 \wedge 1 \wedge B^{n-1}_1) \vee (a_1 \vee b_1)}}} \\ \| \\ \boxed{\begin{array}{c} \left(\boxed{=\dfrac{\bar b_n}{1 \wedge \bar b_n}} \vee \left(\left(\boxed{=\dfrac{b_n}{0 \vee b_n}} \wedge \bar a_{n-1} \wedge \boxed{=\dfrac{b_n}{0 \vee b_n}} \wedge \bar b_{n-1}\right) \vee \cdots \vee \left(\boxed{=\dfrac{b_n}{0 \vee b_n}} \wedge A^{n-1}_1 \wedge \boxed{=\dfrac{b_n}{0 \vee b_n}} \wedge B^{n-1}_1\right)\right) \vee (a_1 \vee b_1)\right) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \ell_{a_n} S_n \end{array}} \end{array}} \; a_n \; \boxed{\begin{array}{c} S_{n-1} \\ \psi \, \| \\ q_{a_n} S_n \end{array}} \\ \| \\ S_n \end{array}$$

where:

$$\varphi \equiv 1 \vee \boxed{\begin{array}{c} 0 \\ \| \\ (0 \wedge \bar a_{n-1} \wedge 0 \wedge \bar b_{n-1}) \vee ((0 \wedge A^{n-1}_{n-2} \wedge 0 \wedge B^{n-1}_{n-2}) \vee \cdots \vee (0 \wedge A^{n-1}_1 \wedge 0 \wedge B^{n-1}_1)) \vee (a_1 \vee b_1) \end{array}}$$

$$\psi \equiv \boxed{=\dfrac{(\bar a_{n-1} \wedge \bar b_{n-1}) \vee ((A^{n-1}_{n-2} \wedge B^{n-1}_{n-2}) \vee \cdots \vee (A^{n-1}_1 \wedge B^{n-1}_1)) \vee (a_1 \vee b_1)}{\left(0 \wedge \boxed{\begin{array}{c} 0 \\ \| \\ \bar b_n \end{array}}\right) \vee \left(\left(\left(1 \vee \boxed{\begin{array}{c} 0 \\ \| \\ b_n \end{array}}\right) \wedge \bar a_{n-1} \wedge \left(1 \vee \boxed{\begin{array}{c} 0 \\ \| \\ b_n \end{array}}\right) \wedge \bar b_{n-1}\right) \vee \cdots \vee \left(\left(1 \vee \boxed{\begin{array}{c} 0 \\ \| \\ b_n \end{array}}\right) \wedge A^{n-1}_1 \wedge \left(1 \vee \boxed{\begin{array}{c} 0 \\ \| \\ b_n \end{array}}\right) \wedge B^{n-1}_1\right)\right) \vee (a_1 \vee b_1)}}$$

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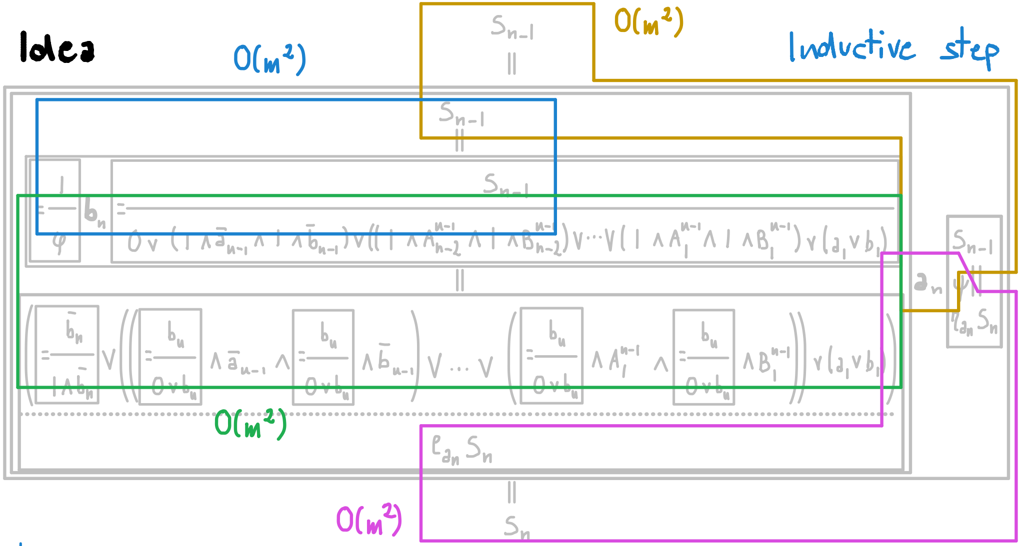
## Idea



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$$\psi \equiv \boxed{= \frac{(\bar{a}_{n-1} \wedge \bar{b}_{n-1}) \vee ((A^{n-1}_{n-2} \wedge B^{n-1}_{n-2}) \vee \cdots \vee (A^{n-1}_1 \wedge B^{n-1}_1)) \vee (a_1 \vee b_1)}{\left(0 \wedge \boxed{\begin{array}{c}0\\ \| \\ \bar{b}_n\end{array}}\right) \vee \left(\left(\left(1 \vee \boxed{\begin{array}{c}0\\ \| \\ b_n\end{array}}\right) \wedge \bar{a}_{n-1} \wedge \left(1 \vee \boxed{\begin{array}{c}0\\ \| \\ b_n\end{array}}\right) \wedge \bar{b}_{n-1}\right) \vee \cdots \vee \left(\left(1 \vee \boxed{\begin{array}{c}0\\ \| \\ b_n\end{array}}\right) \wedge A^{n-1}_1 \wedge \left(1 \vee \boxed{\begin{array}{c}0\\ \| \\ b_n\end{array}}\right) \wedge B^{n-1}_1\right)\right) \vee (a_1 \vee b_1)}}$$

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$$A^n_k \equiv (a_n \vee b_n) \wedge \cdots \wedge (a_{k+1} \vee b_{k+1}) \wedge \bar{a}_k$$

$$( \equiv (a_n \vee b_n) \wedge A^{n-1}_k \quad \text{if } n-1 > k)$$

$$B^n_k \equiv (a_n \vee b_n) \wedge \cdots \wedge (a_{k+1} \vee b_{k+1}) \wedge \bar{b}_k$$

$$( \equiv (a_n \vee b_n) \wedge B^{n-1}_k \quad \text{if } n-1 > k)$$

## Proof

Implements the idea in the proof system defined in the following.

# CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

**Theorem** Given a proof of $A$, we can build a cut-free proof of $A$.

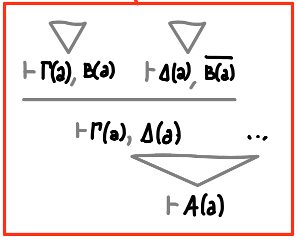
# CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

**Theorem** Given a proof of $A$, we can build a cut-free proof of $A$.

$$\begin{array}{c} 1 \\ \varphi \,\Vert \\ A \end{array}$$

is the given derivation

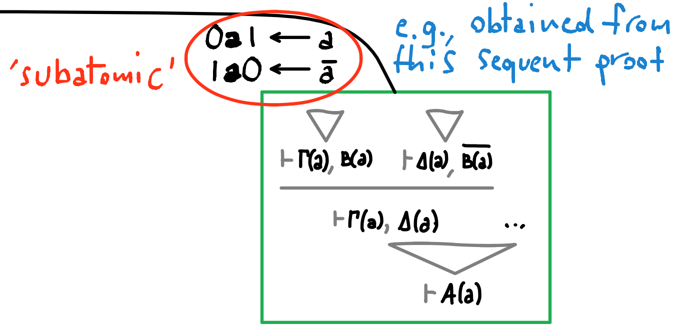
e.g., obtained from this sequent proof



# CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

**Theorem** Given a proof of A, we can build a cut-free proof of A.

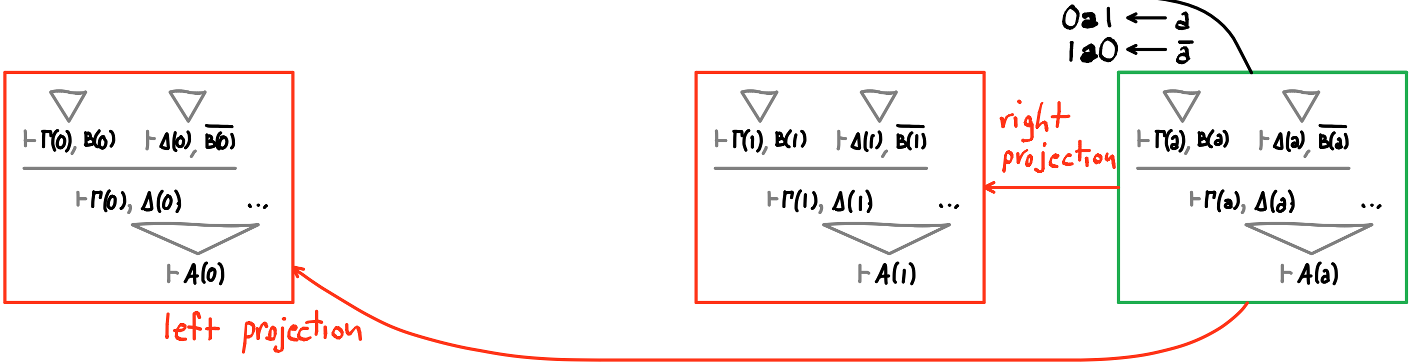
$$\begin{array}{c} 1 \\ \varphi \, \| \\ A \end{array}$$ is the given derivation and a an atom appearing in a cut instance



# CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

**Theorem** Given a proof of A, we can build a cut-free proof of A.

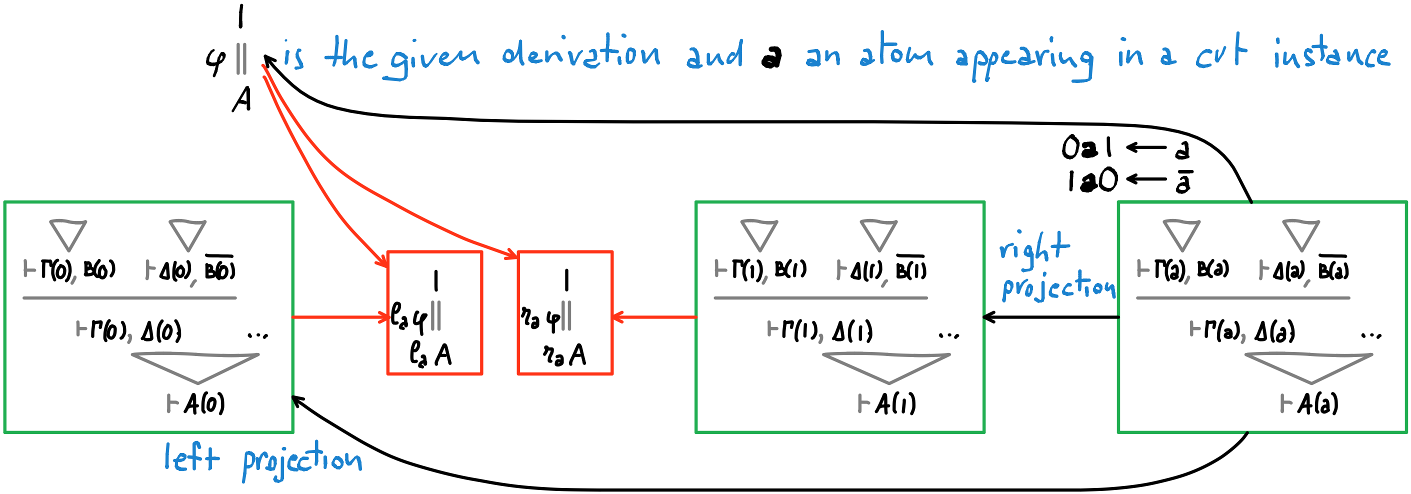
$\varphi \, \overset{1}{\underset{A}{\Vert}}$ is the given derivation and **a** an atom appearing in a cut instance



# CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

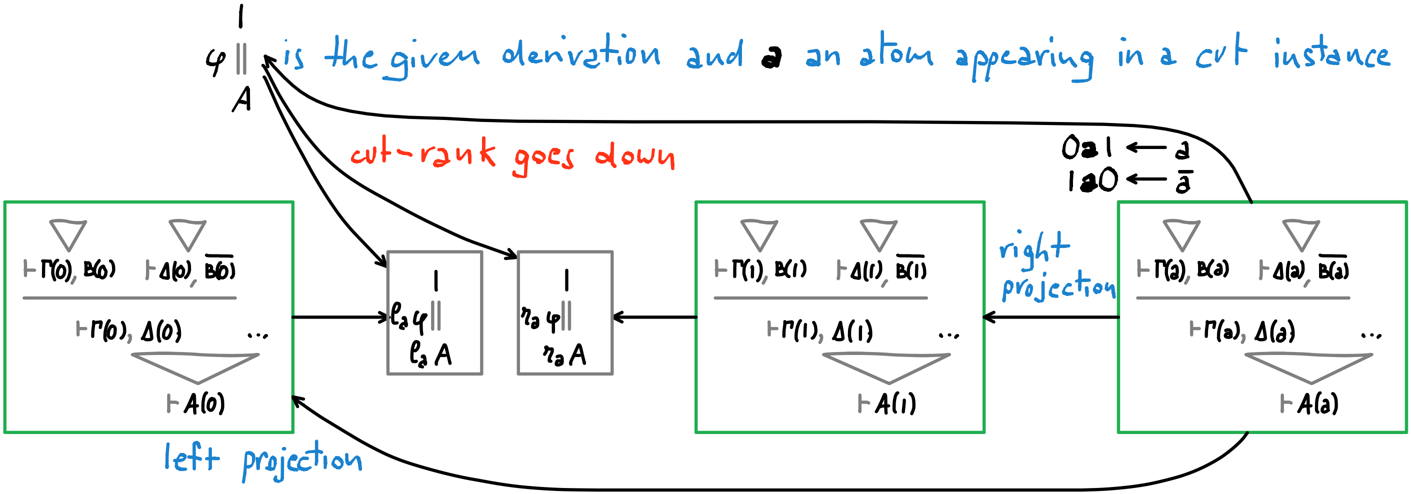
**Theorem** Given a proof of $A$, we can build a cut-free proof of $A$.

$\begin{matrix} I \\ \varphi \| \\ A \end{matrix}$ is the given derivation and **a** an atom appearing in a cut instance



# CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

**Theorem** Given a proof of $A$, we can build a cut-free proof of $A$.



$\overset{1}{\phi \Vert A}$ is the given derivation and $a$ an atom appearing in a cut instance

# CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

**Theorem** Given a proof of $A$, we can build a cut-free proof of $A$.

$$\begin{array}{c} 1 \\ \varphi \Big\| \\ A \end{array}$$

is the given derivation and $\mathbf{a}$ an atom appearing in a cut instance

cut-rank goes down

$$\boxed{= \frac{1}{\boxed{\begin{array}{c} 1 \\ \ell_a \varphi \Big\| \\ \ell_a A \end{array}} \; \mathbf{a} \; \boxed{\begin{array}{c} 1 \\ r_a \varphi \Big\| \\ r_a A \end{array}}}}$$

decision trees:

$$(1\,\mathbf{a}\,0)\,\mathbf{b}\,1 = \begin{cases} 1 & \text{if } b \\ \bar{a} & \text{if } \bar{b} \end{cases}$$

# CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

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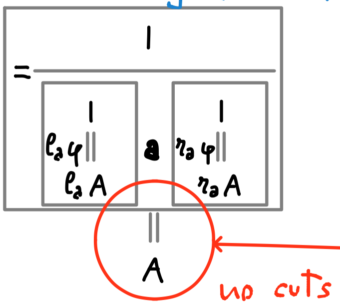


decision trees:

$$(1 \mathbf{a} 0)\, \mathbf{b}\, 1 = \begin{cases} 1 & \text{if } b \\ \bar{a} & \text{if } \bar{b} \end{cases}$$

repeated applications + contractions

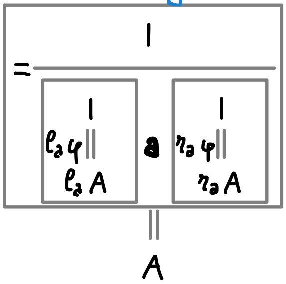
$$\frac{(A \beta B)\, \mathbf{a}\, (C \beta D)}{(A \mathbf{a} C)\, \beta\, (B \mathbf{a} D)}$$

# CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

**Theorem** Given a proof of $A$, we can build a cut-free proof of $A$.

$\begin{array}{c}1\\ \varphi\,\|\\ A\end{array}$ is the given derivation and $\mathbf{a}$ an atom appearing in a cut instance

cut-rank goes down



decision trees:

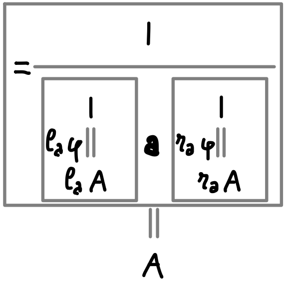
$$(1\,\mathbf{a}\,0)\,\mathbf{b}\,1 = \begin{cases} 1 & \text{if } b \\ \bar{a} & \text{if } \bar{b} \end{cases}$$

This is free of cuts in $\mathbf{a}$. Repeat.

# CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

**Theorem** Given a proof of $A$, we can build a cut-free proof of $A$.

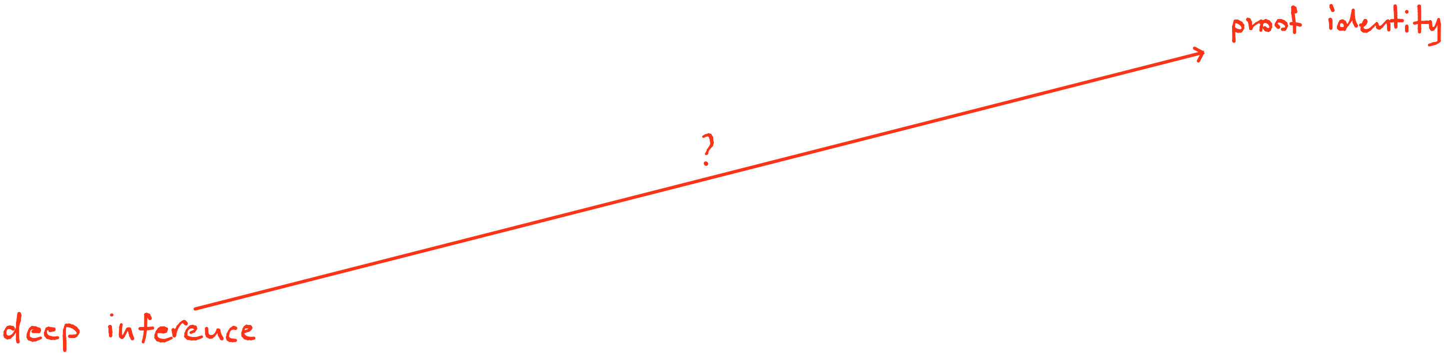
**Proof** If $\begin{array}{c} I \\ \varphi \, \| \\ A \end{array}$ is the given derivation and $\mathbf{a}$ an atom appearing in a cut instance, build



.

This is free of cuts in $\mathbf{a}$. Repeat.

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



proof identity

?

deep inference

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS

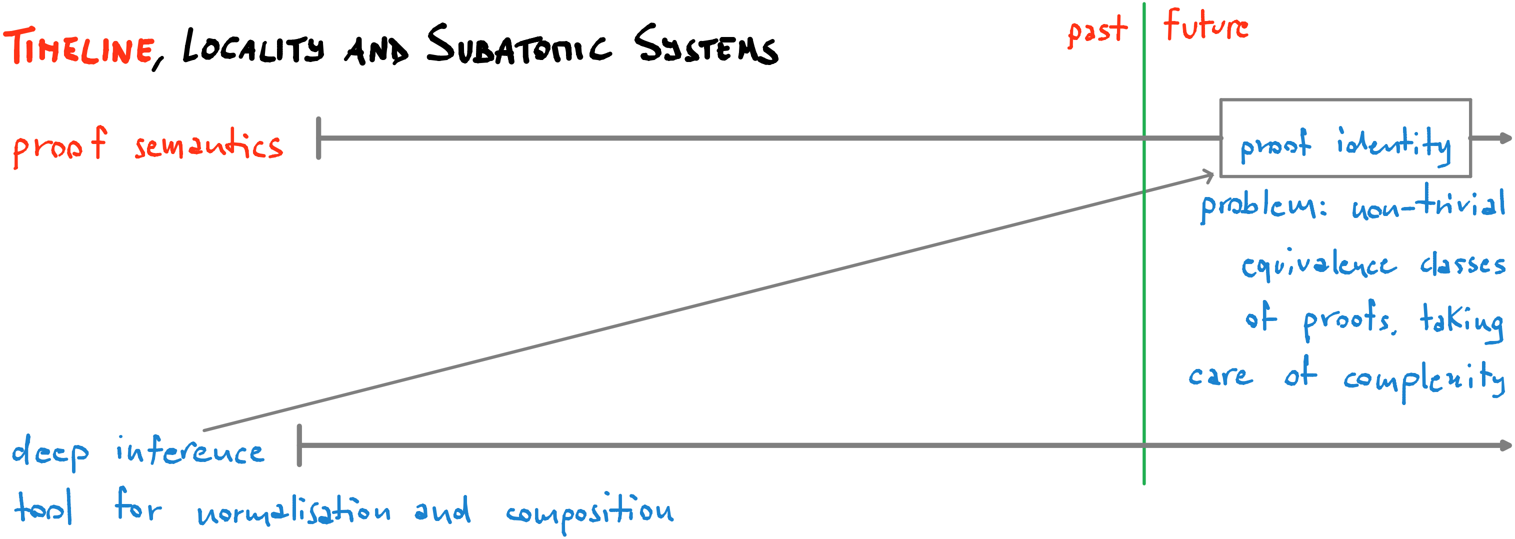
proof identity

problem: non-trivial
equivalence classes
of proofs, taking
care of complexity

deep inference

tool for normalisation and composition

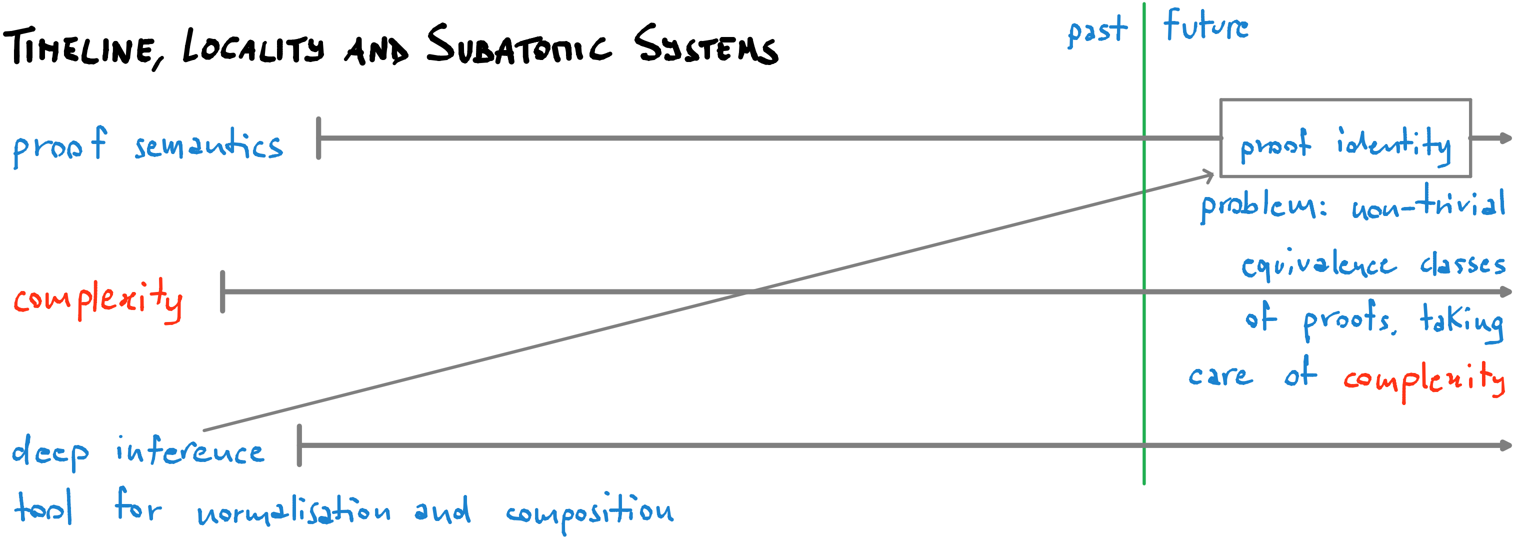
# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



problem: non-trivial equivalence classes of proofs, taking care of complexity

tool for normalisation and composition

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



past

future

proof semantics

proof identity

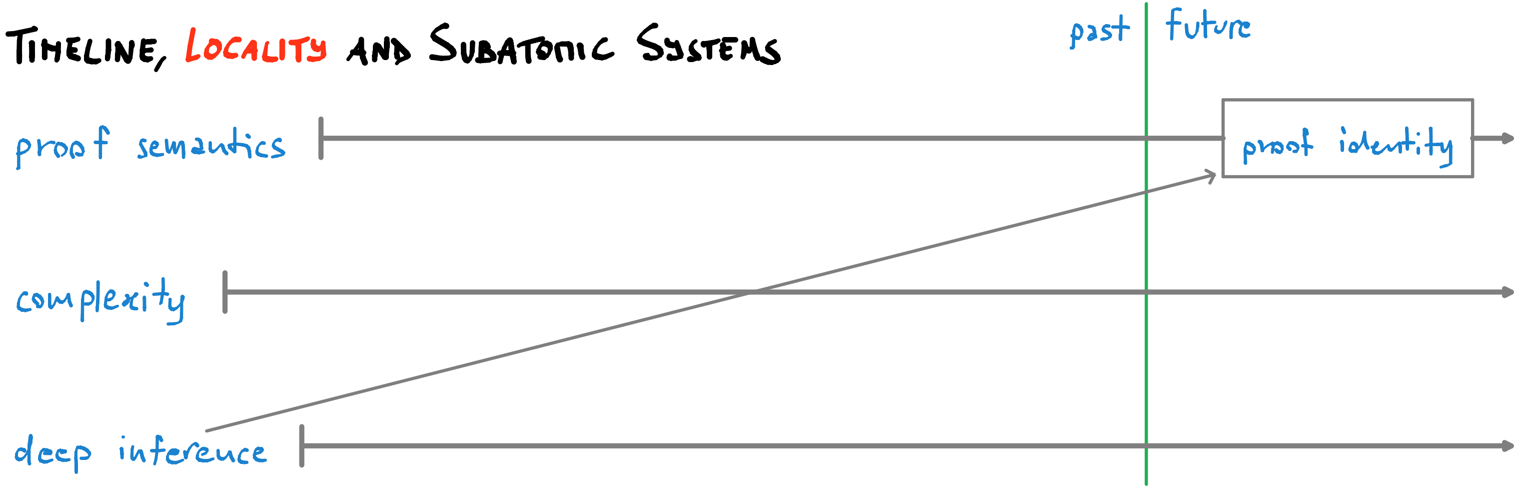
problem: non-trivial equivalence classes of proofs, taking care of complexity

complexity

deep inference

tool for normalisation and composition

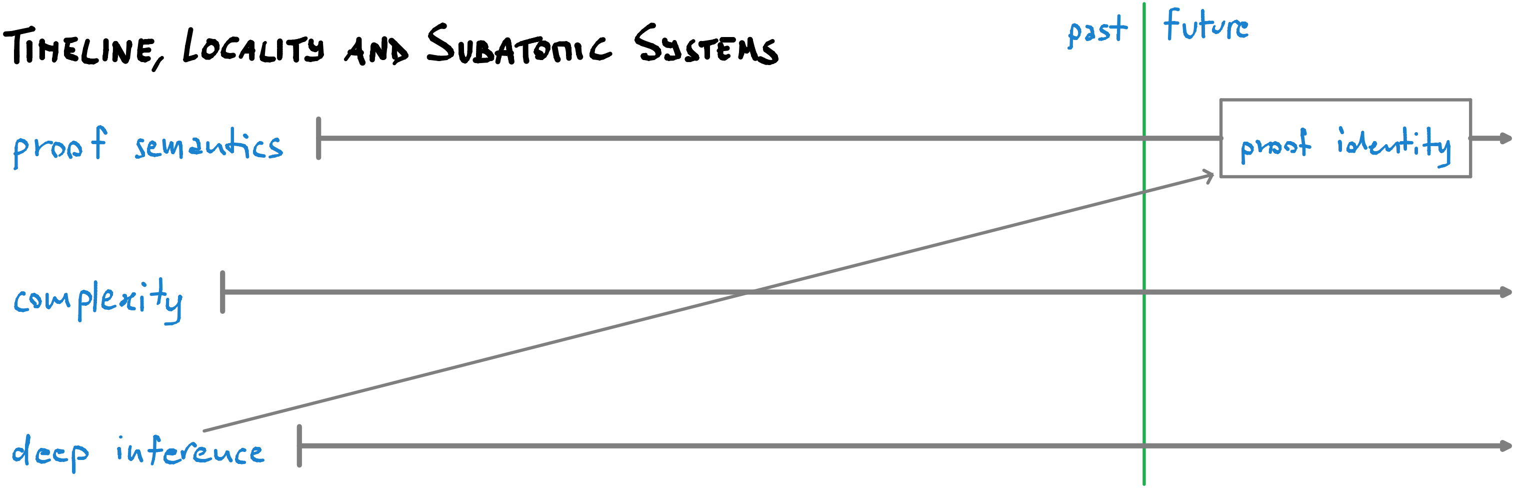
# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



$$\text{cut}\,\frac{\vdash \Gamma, A \qquad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

Can we check an instance of this Gentzen cut in constant time?

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



past

future

proof semantics

proof identity

complexity

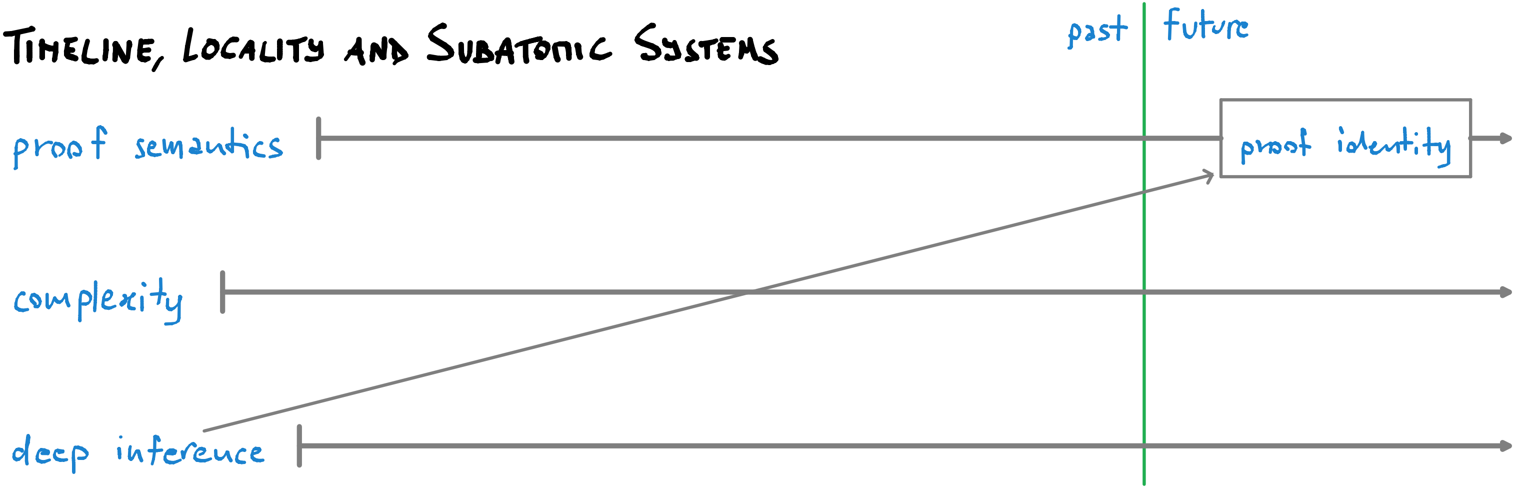
deep inference

$$\text{cut}\,\frac{\vdash \Gamma, A \qquad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

Can we check an instance of this Gentzen cut in constant time? No.

1: negation, $A$ is unbounded

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



$$\text{cut}\,\frac{\vdash \Gamma, A \qquad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

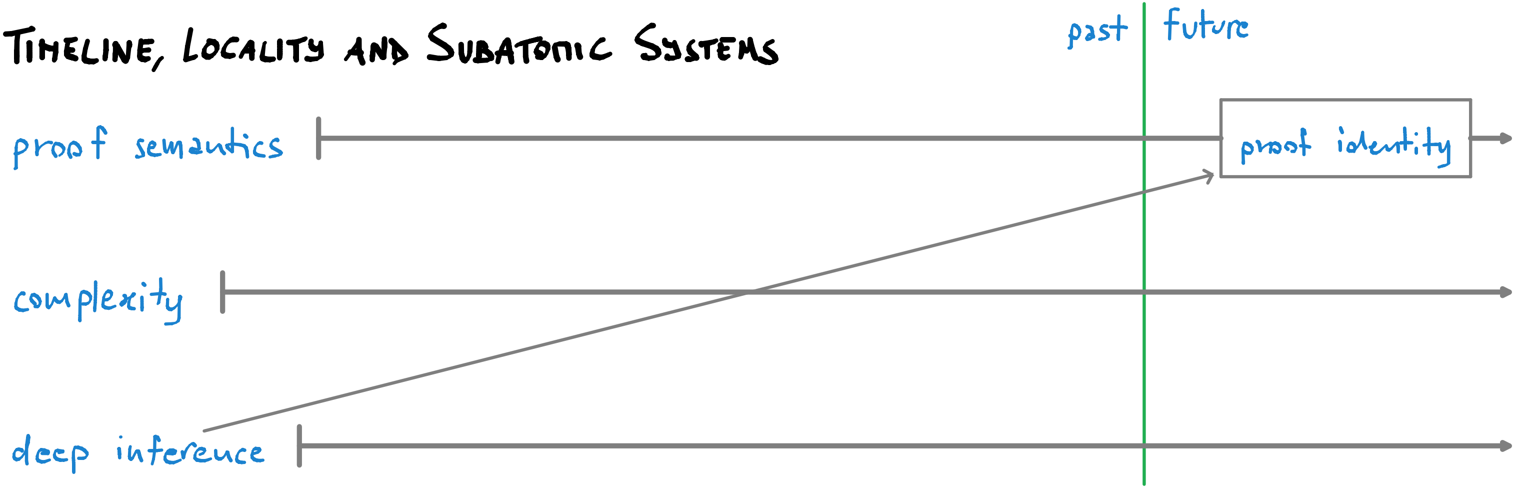
Can we check an instance of this Gentzen cut in constant time? No.

1: negation, $A$ is unbounded;

2: contraction, $\Gamma$ is unbounded.

We say that the cut is non-local.

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



non-local

Gentzen

$$\text{cut}\,\frac{\vdash \Gamma, A \qquad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

Can we check an instance of this Gentzen cut in constant time? No.

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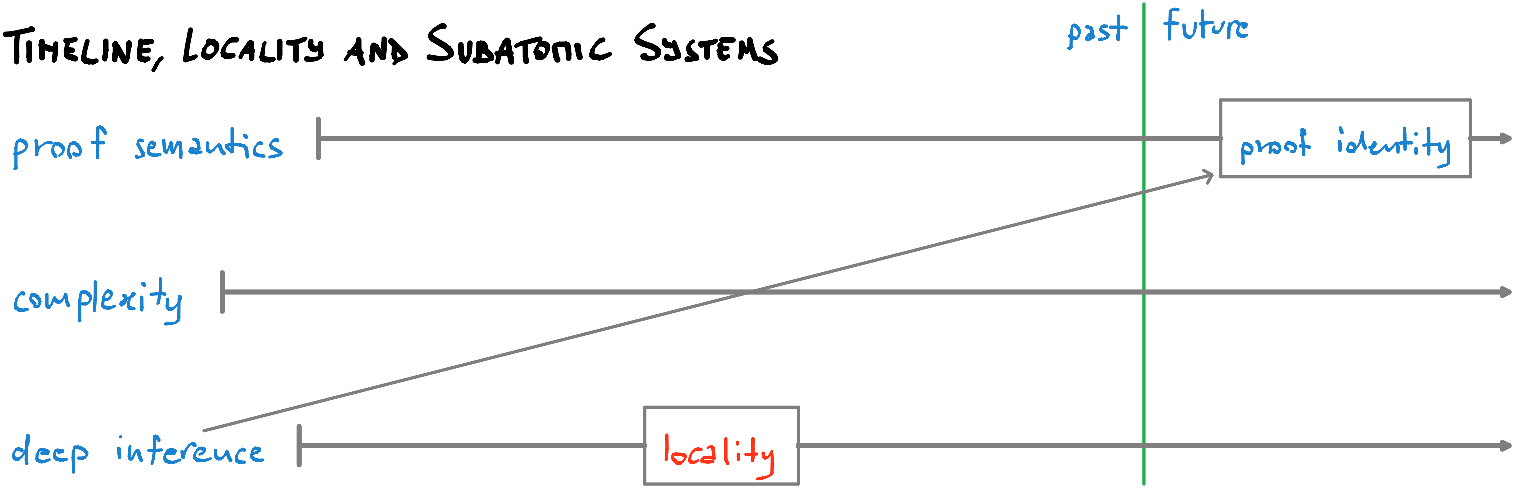
# Timeline, Locality and Subatomic Systems



non-local

Gentzen

$$\mathsf{cut}\,\frac{\vdash \Gamma, A \qquad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

linear | atomic

$$\mathsf{switch}\,\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

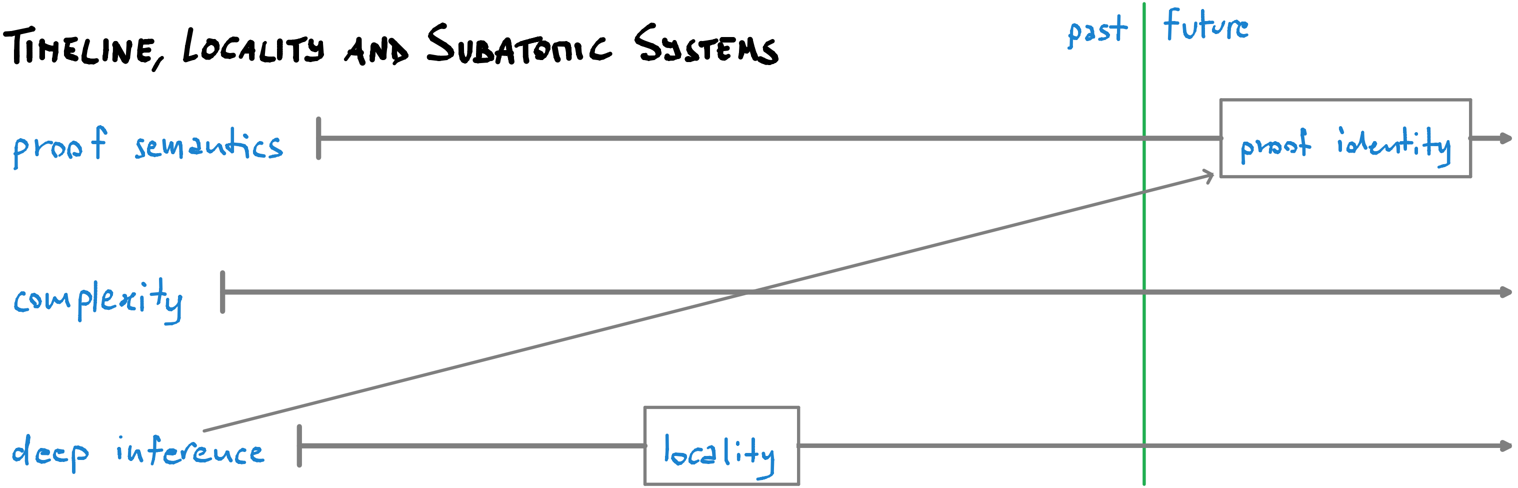
$$\mathsf{at\ cut}\,\frac{a \wedge \bar{a}}{0}$$

linear or atomic = local

negation, $A$ is unbounded

contraction, $\Gamma$ is unbounded

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



non-local

Gentzen

$$\text{cut}\ \frac{\vdash \Gamma, A \qquad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

linear

$$\text{switch}\ \frac{\overbrace{(A_1 \vee A_2)}^{\overset{A}{\equiv}} \wedge (\bar{A}_1 \wedge \bar{A}_2)}{(A_1 \wedge \bar{A}_1) \vee (A_2 \wedge \bar{A}_2)}$$

cut formulae are reduced

atomic

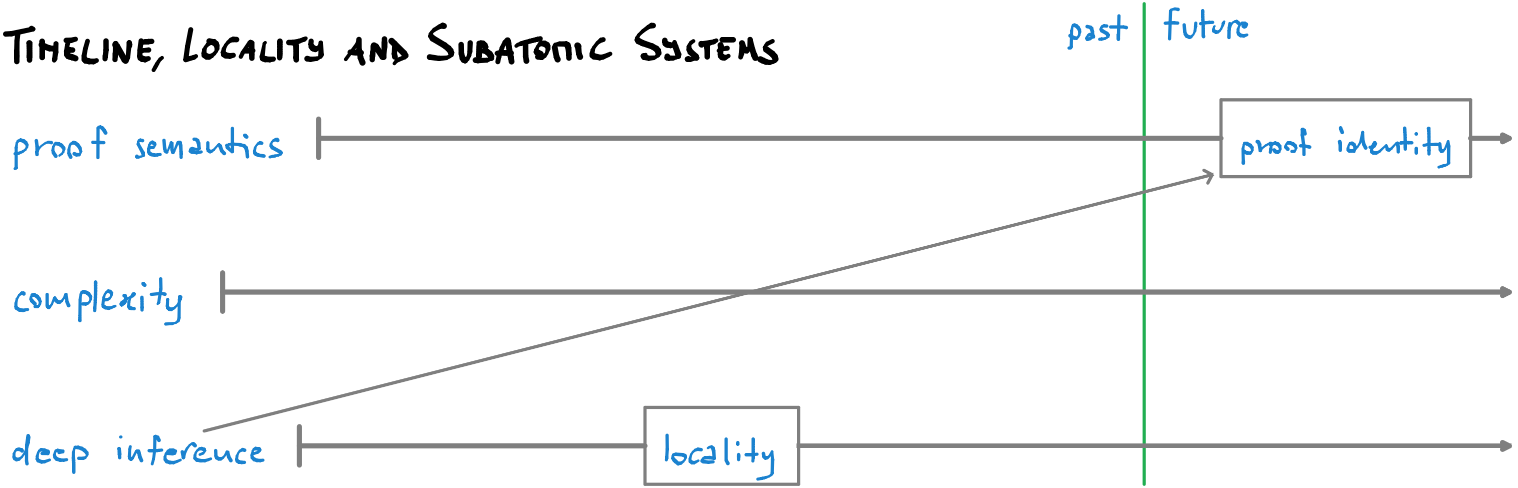
$$\text{at. cut}\ \frac{a \wedge \bar{a}}{0}$$

linear or atomic = local

negation, $A$ is unbounded

contraction, $\Gamma$ is unbounded

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



non-local

Gentzen

$$\text{cut}\,\frac{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

linear | atomic

$$\text{switch}\,\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \text{at. cut}\,\frac{a \wedge \bar{a}}{0}$$

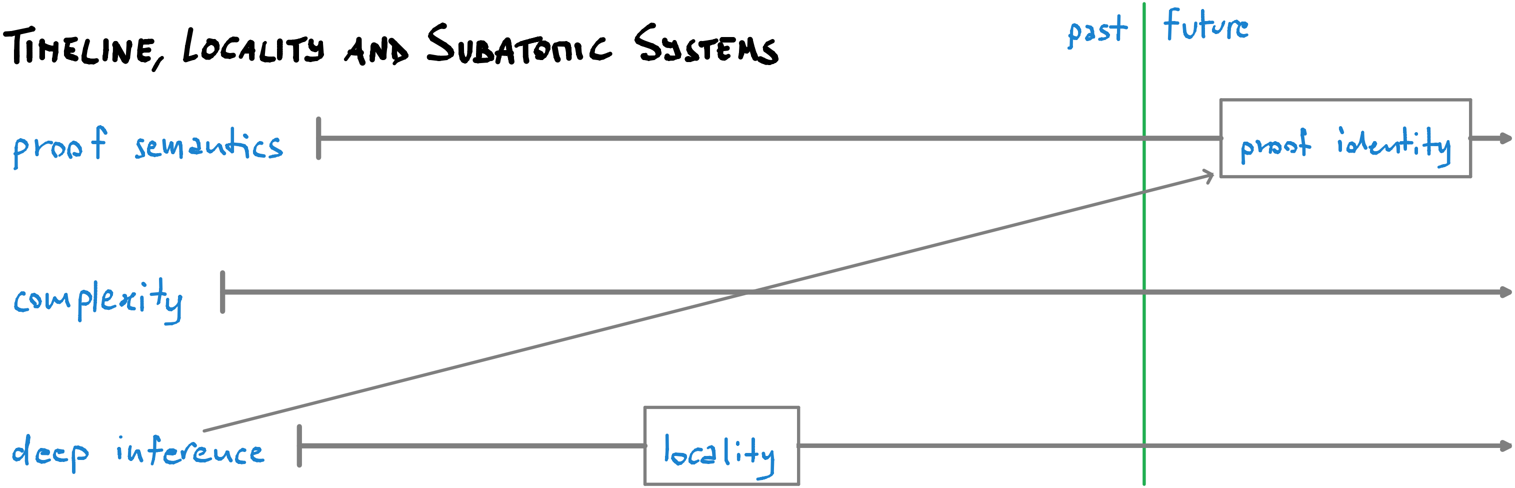
$$\text{medial}\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \text{at. contr.}\,\frac{a \vee a}{a}$$

linear or atomic = local

negation, $A$ is unbounded

contraction, $\Gamma$ is unbounded

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



past | future

proof semantics → proof identity

complexity

deep inference → locality

non-local

Gentzen

$$\mathrm{cut}\,\frac{\vdash \Gamma, A \qquad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

contractions are reduced

linear | atomic

$$\mathrm{switch}\,\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \mathrm{at.\ cut}\,\frac{a \wedge \bar{a}}{0}$$

$$\mathrm{medial}\,\frac{(\Gamma_1 \wedge \Gamma_2) \vee (\Gamma_1 \wedge \Gamma_2)}{(\Gamma_1 \vee \Gamma_1) \wedge (\Gamma_2 \vee \Gamma_2)} \qquad \mathrm{at.\ contr.}\,\frac{a \vee a}{a}$$

linear or atomic = local

negation, $A$ is unbounded

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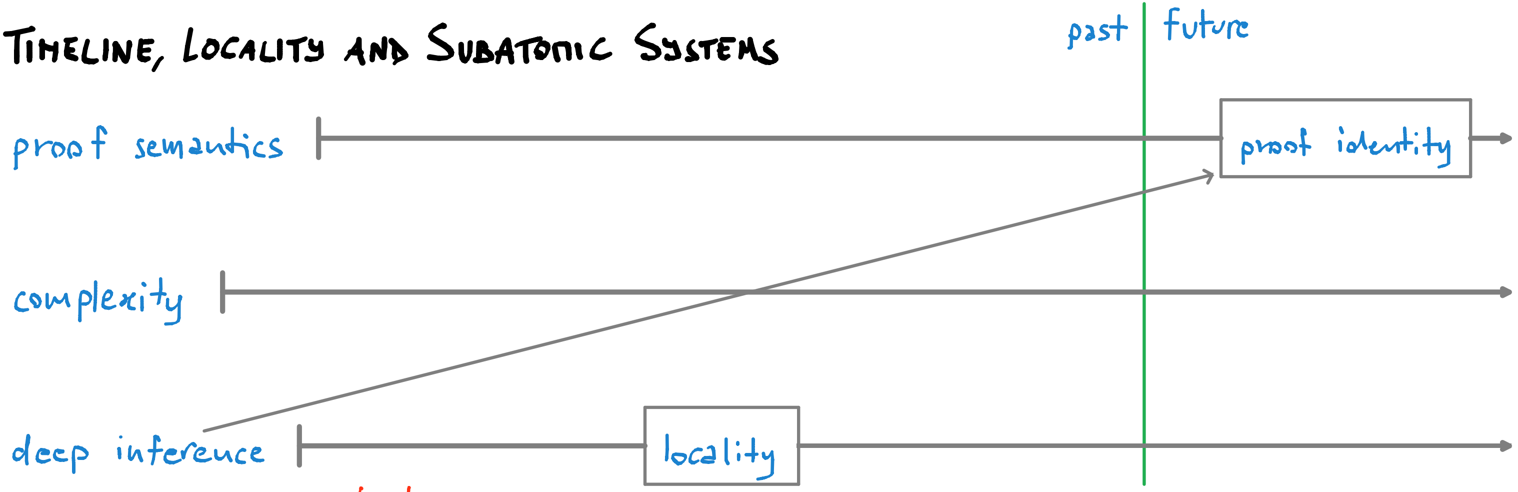
# Timeline, Locality and Subatomic Systems



non-local

**Gentzen**

$$\text{cut}\;\frac{\vdash \Gamma, A \qquad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

←

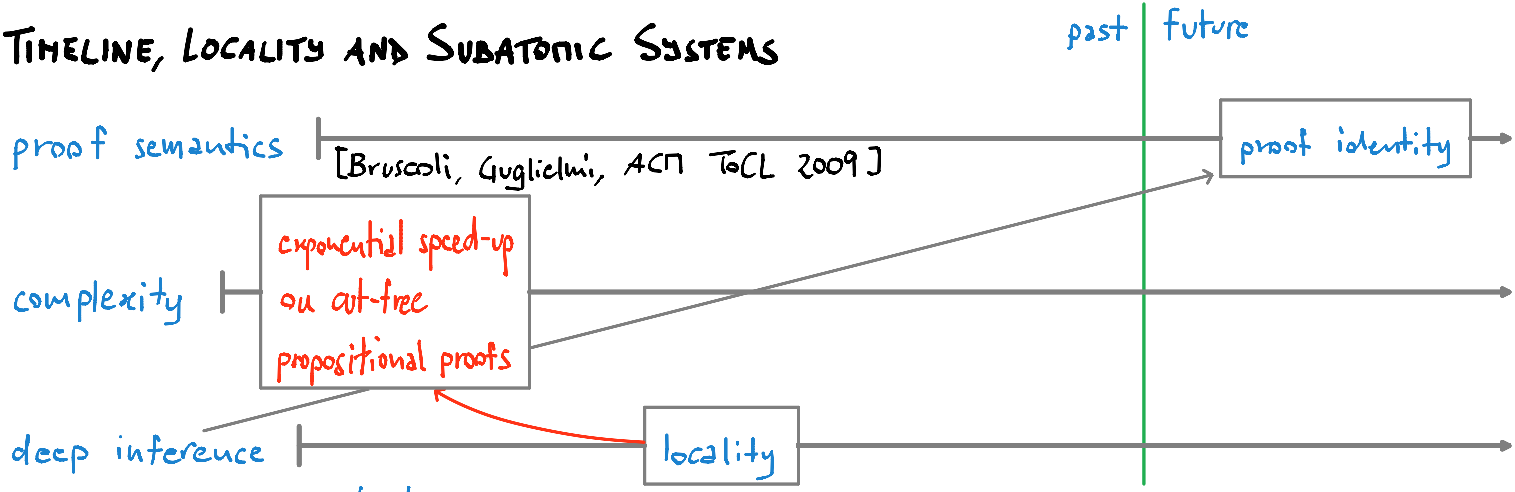
local

$$\text{switch}\;\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \text{at. cut}\;\frac{a \wedge \bar{a}}{0}$$

$$\text{medial}\;\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \text{at. contr.}\;\frac{a \vee a}{a}$$

deep inference

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



past | future

proof semantics

[Bruscoli, Guglielmi, ACM ToCL 2009]

proof identity

complexity

exponential speed-up on cut-free propositional proofs

deep inference

locality

non-local

Gentzen

$$\text{cut}\ \frac{\vdash \Gamma, A \qquad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

←

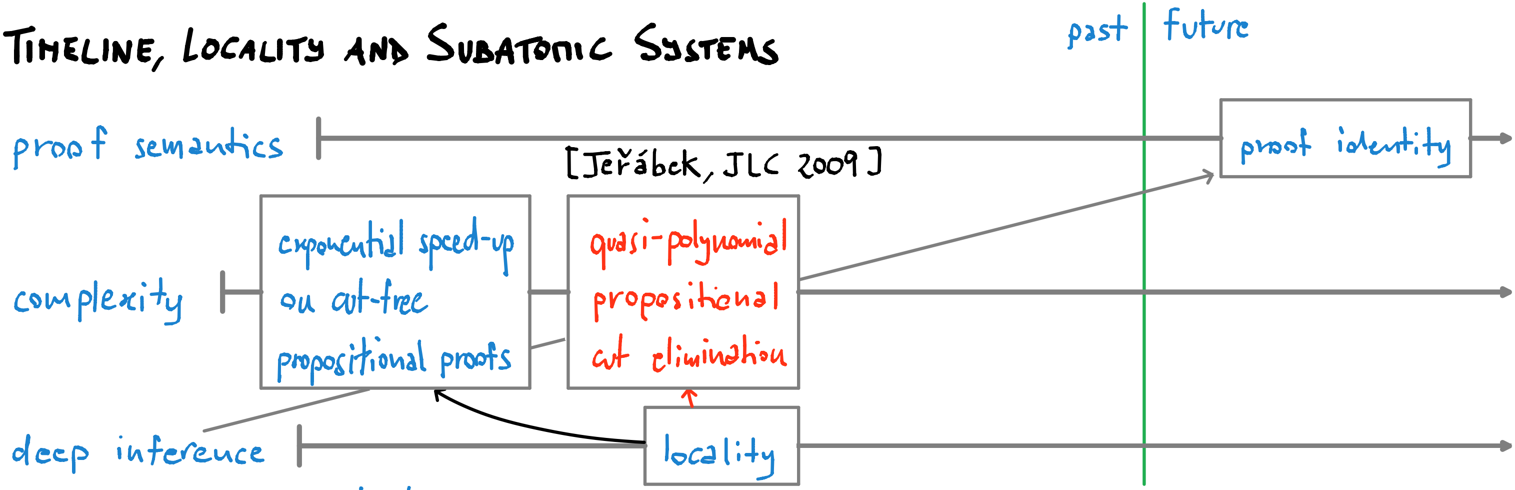
local

deep inference

$$\text{switch}\ \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \text{at. cut}\ \frac{a \wedge \bar{a}}{0}$$

$$\text{medial}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \text{at. contr.}\ \frac{a \vee a}{a}$$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



non-local

Gentzen

$$\text{cut}\;\frac{\vdash \Gamma, A \qquad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

←

local

deep inference

$$\text{switch}\;\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \text{at. cut}\;\frac{a \wedge \bar{a}}{0}$$

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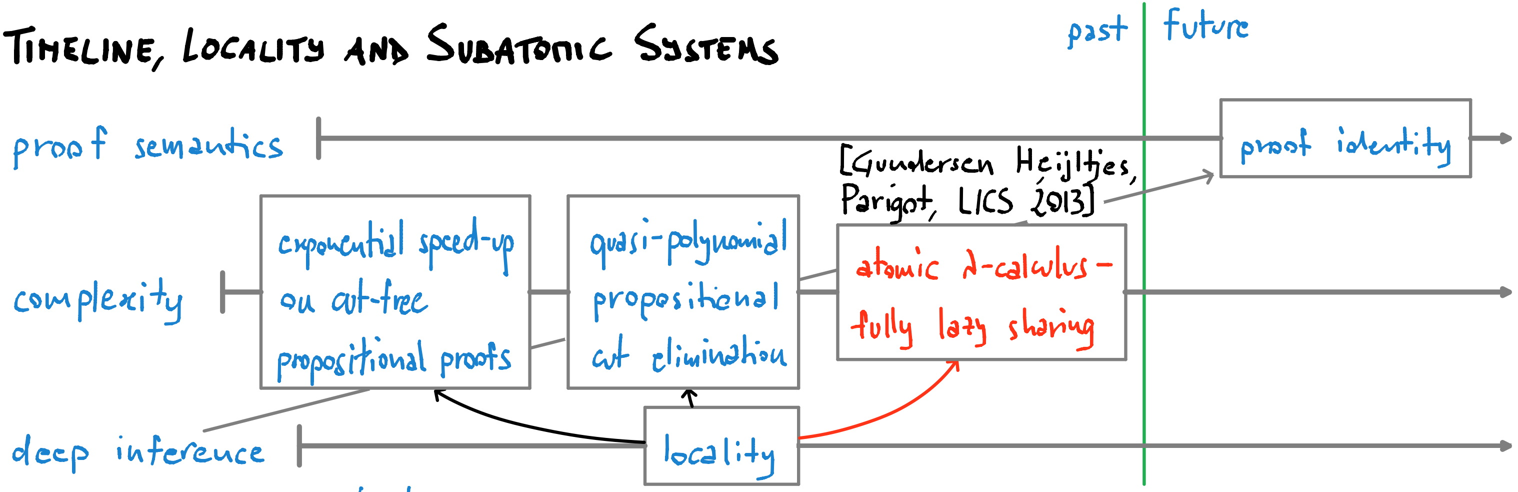
# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



past

future

proof semantics

proof identity

[Gundersen Heijltjes, Parigot, LICS 2013]

complexity

exponential speed-up on cut-free propositional proofs

quasi-polynomial propositional cut elimination

atomic λ-calculus – fully lazy sharing

deep inference

locality

non-local

Gentzen

$$\text{cut}\ \frac{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

←

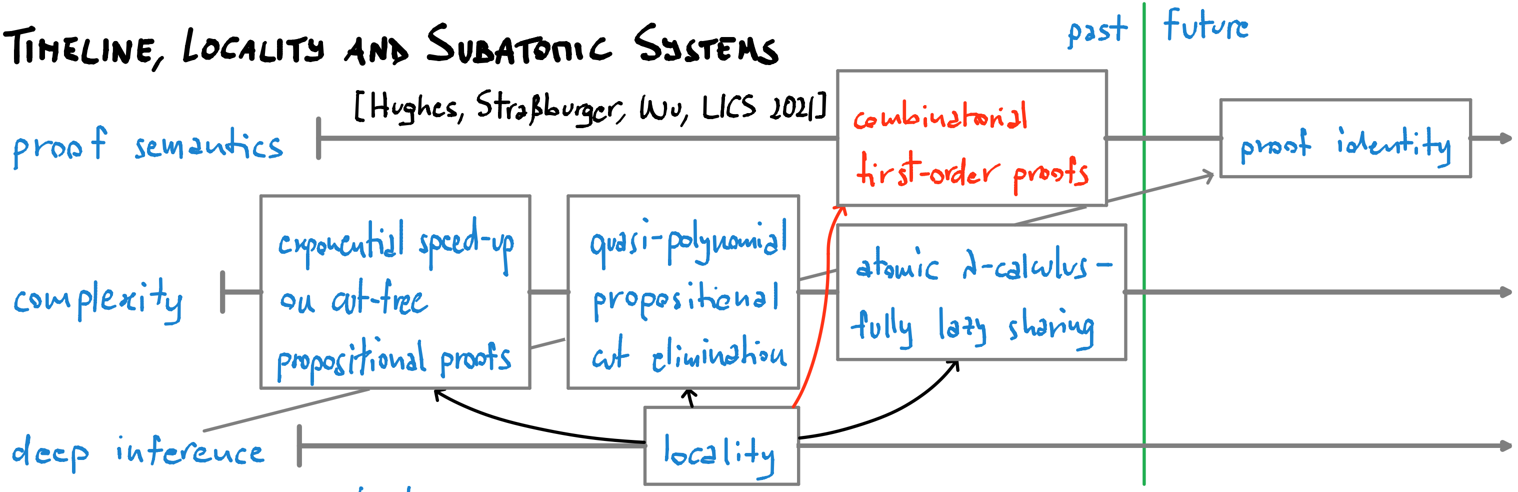
local

$$\text{switch}\ \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \text{at. cut}\ \frac{a \wedge \bar{a}}{0}$$

$$\text{medial}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \text{at. contr.}\ \frac{a \vee a}{a}$$

deep inference

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



non-local

Gentzen

$$\text{cut}\ \frac{\vdash \Gamma, A \qquad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

←

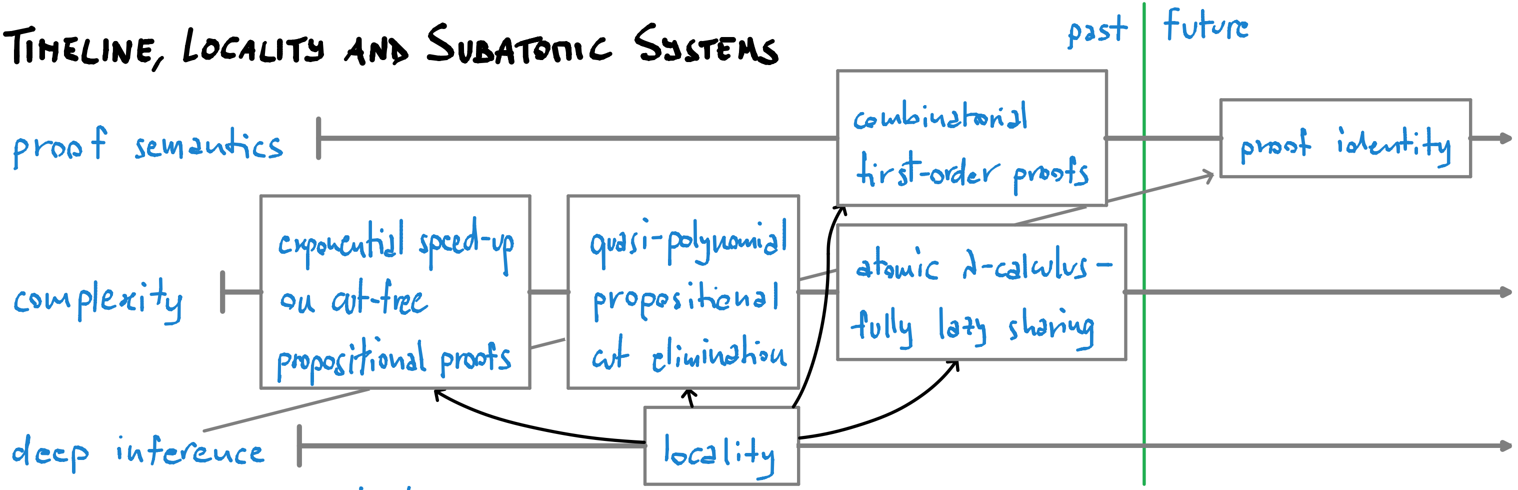
local

$$\text{switch}\ \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \text{at. cut}\ \frac{a \wedge \bar{a}}{0}$$

$$\text{medial}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \text{at. contr.}\ \frac{a \vee a}{a}$$

deep inference

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



non-local

Gentzen

$$\xcancel{\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, \bar{A}}{\vdash \Gamma}\,\text{cut}}$$

←

local

$$\text{switch}\ \dfrac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \text{at. cut}\ \dfrac{a \wedge \bar{a}}{0}$$

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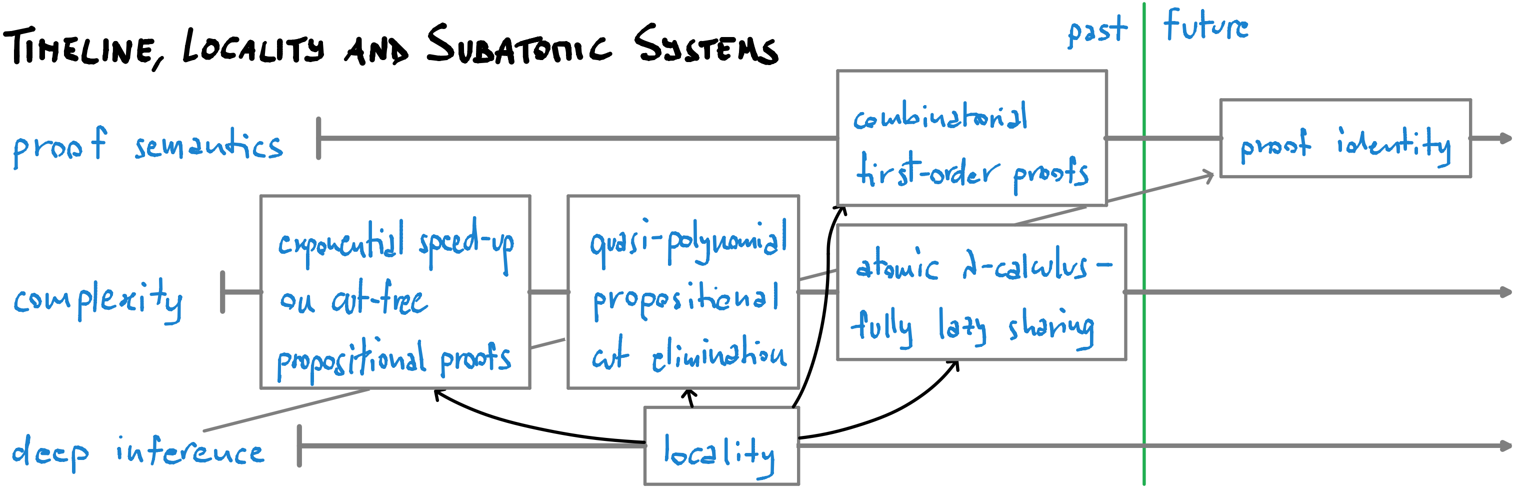
deep inference

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



past | future

proof semantics

complexity

deep inference

exponential speed-up on cut-free propositional proofs

quasi-polynomial propositional cut elimination

combinatorial first-order proofs

atomic λ-calculus – fully lazy sharing

proof identity

locality

instances of the shape

$$\mathsf{switch}\ \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \text{at. cut}\ \frac{a \wedge \bar{a}}{0}$$

$$\mathsf{medial}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \text{at. contr.}\ \frac{a \vee a}{a}$$

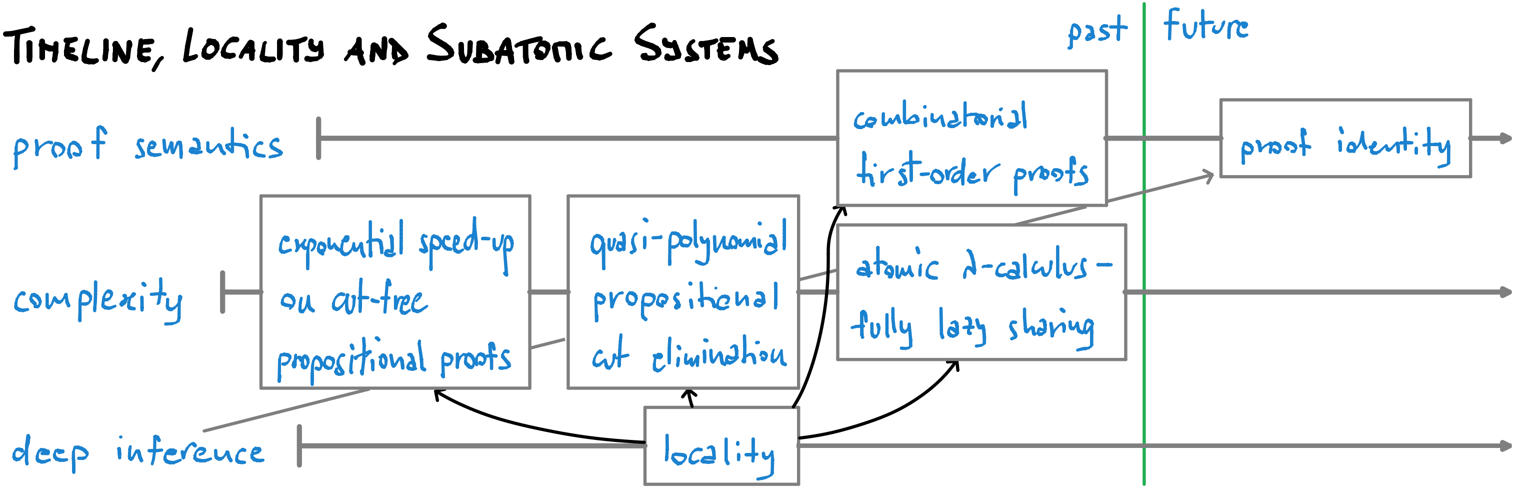
shape

saturation

$$\frac{(A \,\beta\, B) \,\alpha\, (C \,\hat{\beta}\, D)}{(A \,\alpha\, C) \,\beta\, (B \,\alpha\, D)} \qquad \hat{\vee} = \hat{\wedge} = \wedge$$

$$\alpha, \beta \in \{\vee, \wedge \qquad \}$$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



instances of the shape

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all instances are sound

shape

$$\frac{(A \,\beta\, B)\ \alpha\ (C \,\beta\, D)}{(A \,\alpha\, C)\ \beta\ (B \,\check{\alpha}\, D)}$$

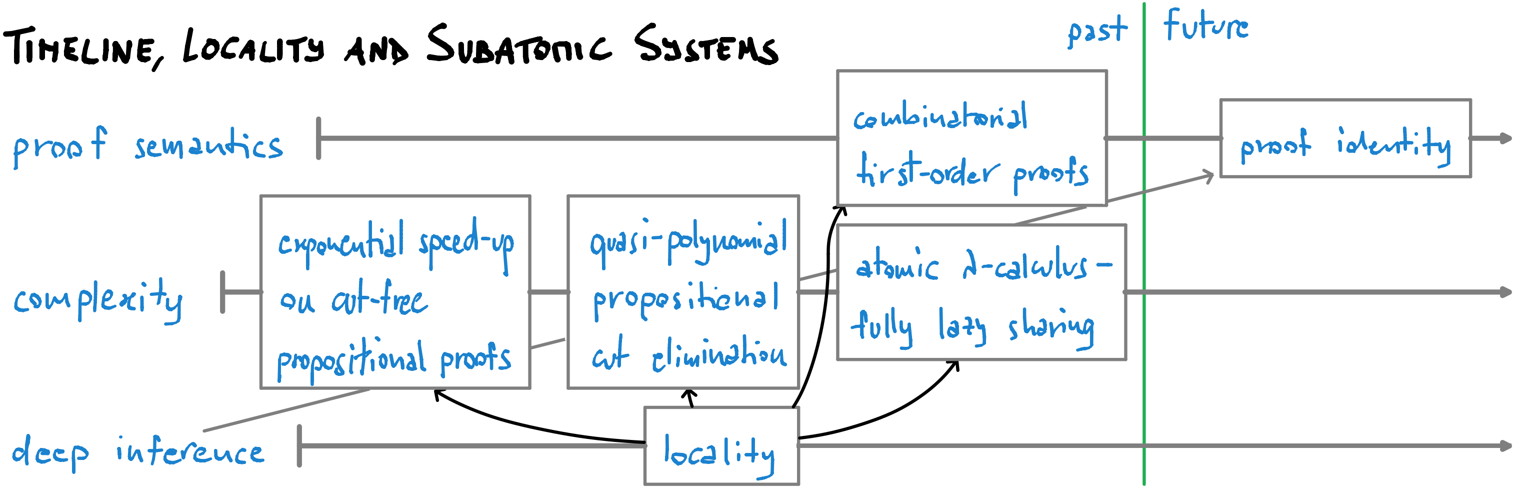
saturation

$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\alpha, \beta \in \{\vee, \wedge \qquad \}$$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS


past
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proof semantics
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deep inference
locality

instances of the shape

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$$\text{medial}\ \frac{(A\wedge B)\vee(C\wedge D)}{(A\vee C)\wedge(B\vee D)} \qquad \text{at. contr.}\ \frac{a\vee a}{a}$$

all instances are sound

shape

$$\frac{(A\,\beta\,B)\,\alpha\,(C\,\beta\,D)}{(A\,\check{\alpha}\,C)\,\beta\,(B\,\alpha\,D)}$$

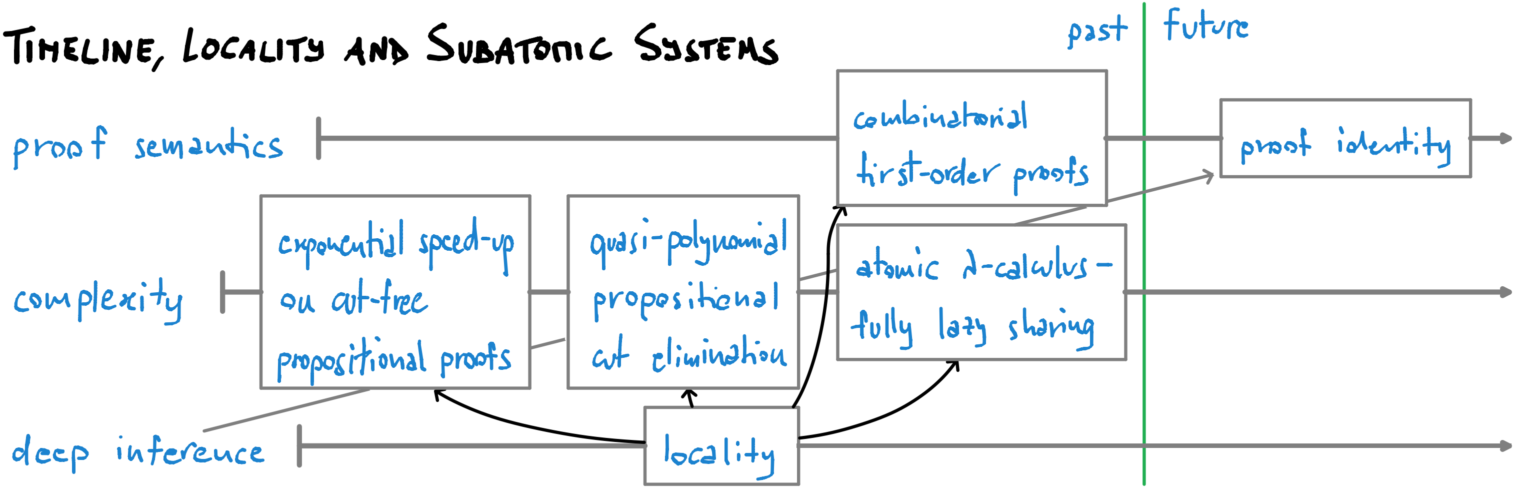
saturation

$$\check{\vee}=\check{\wedge}=\vee$$

$$\hat{\vee}=\hat{\wedge}=\wedge$$

$$\alpha,\beta\in\{\vee,\wedge \qquad\qquad \}$$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



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instances of the shape

$$\text{switch}\ \frac{(A \wedge B) \wedge (C \vee D)}{(A \wedge C) \vee (B \wedge D)} \qquad \text{at. cut}\ \frac{a \wedge \bar{a}}{0}$$

$$\text{medial}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \text{at. contr.}\ \frac{a \vee a}{a}$$

all instances are sound

shape

$$\frac{(A\, \hat{\beta}\, B)\ \alpha\ (C\, \beta\, D)}{(A\, \alpha\, C)\ \beta\ (B\, \alpha\, D)}$$

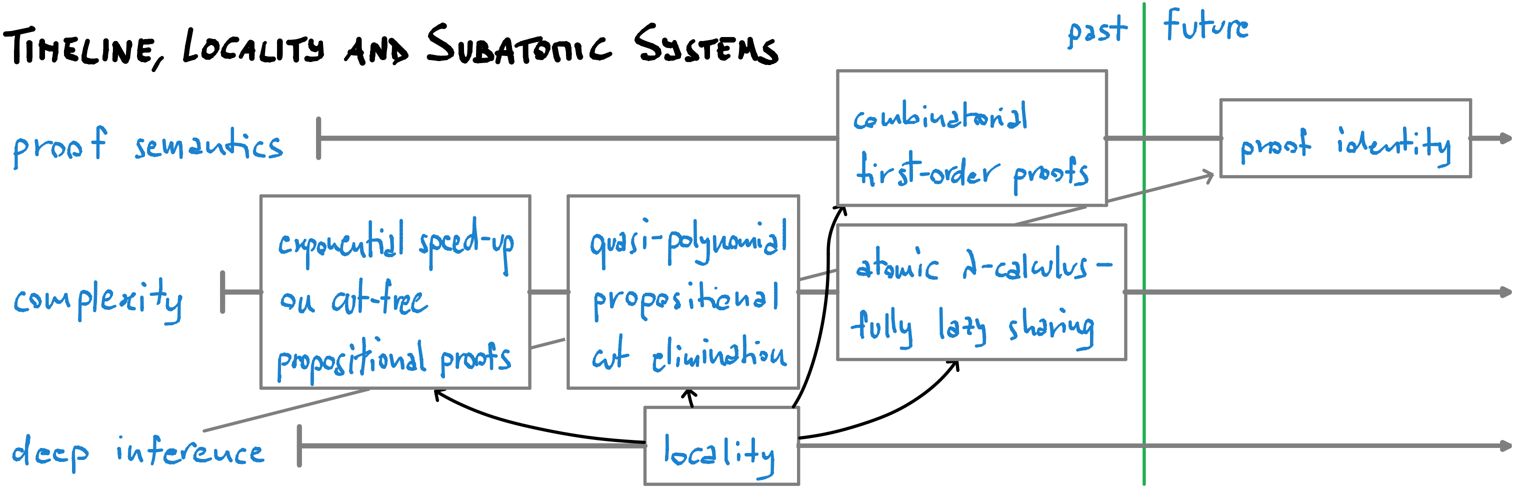
saturation

$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\alpha, \beta \in \{\vee, \wedge \qquad \}$$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



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instances of the shape

switch $\dfrac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$ at. cut $\dfrac{a \wedge \bar{a}}{0}$

medial $\dfrac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$ at. contr. $\dfrac{a \vee a}{a}$

shape

$$\frac{(A \,\dot{\beta}\, B) \;\alpha\; (C \,\dot{\beta}\, D)}{(A \,\dot{\alpha}\, C) \;\beta\; (B \,\dot{\alpha}\, D)}$$

saturation

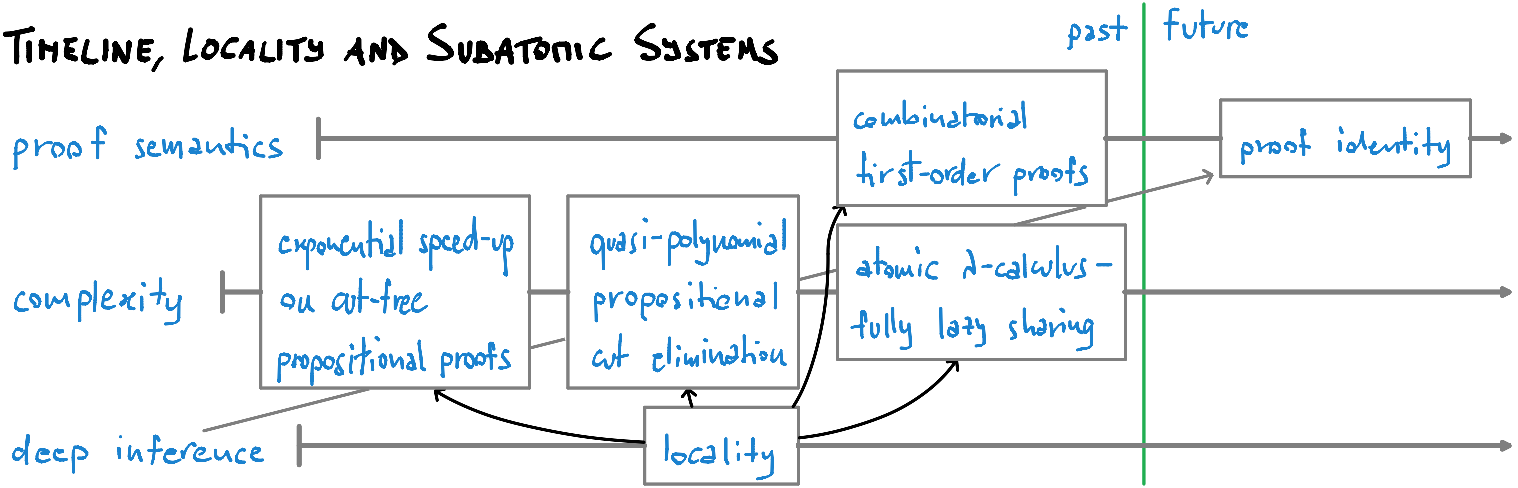
$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

one corner is saturated

$$\alpha, \beta \in \{\vee, \wedge \qquad\qquad\qquad \}$$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



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$$\text{switch}\ \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \text{at. cut}\ \frac{a \wedge \bar{a}}{0}$$

$$\text{medial}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \text{at. contr.}\ \frac{a \vee a}{a}$$

What about atomic rules?

shape

$$\frac{(A\ \dot{\beta}\ B)\ \alpha\ (C\ \dot{\beta}\ D)}{(A\ \dot{\alpha}\ C)\ \beta\ (B\ \dot{\alpha}\ D)}$$

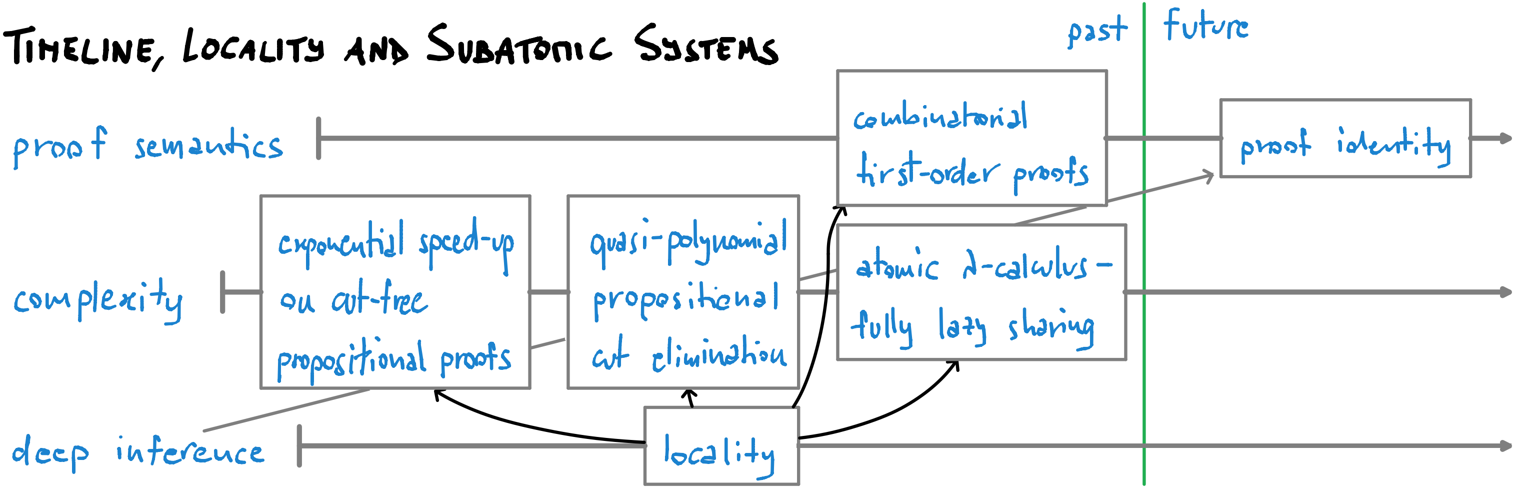
saturation

$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\alpha, \beta \in \{\vee, \wedge \qquad \}$$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



instances of the shape

$$\text{switch}\ \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \text{at. cut}\ \frac{a \wedge \bar{a}}{0}$$

$$\text{medial}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \text{at. contr.}\ \frac{a \vee a}{a}$$

instance of the shape

$$\frac{(A\ a\ B) \wedge (C\ a\ D)}{(A \wedge C)\ a\ (B \wedge D)}$$

shape

$$\frac{(A\ \dot{\beta}\ B)\ \alpha\ (C\ \dot{\beta}\ D)}{(A\ \dot{\alpha}\ C)\ \beta\ (B\ \dot{\alpha}\ D)}$$

saturation

$\check{\vee} = \check{\wedge} = \vee$

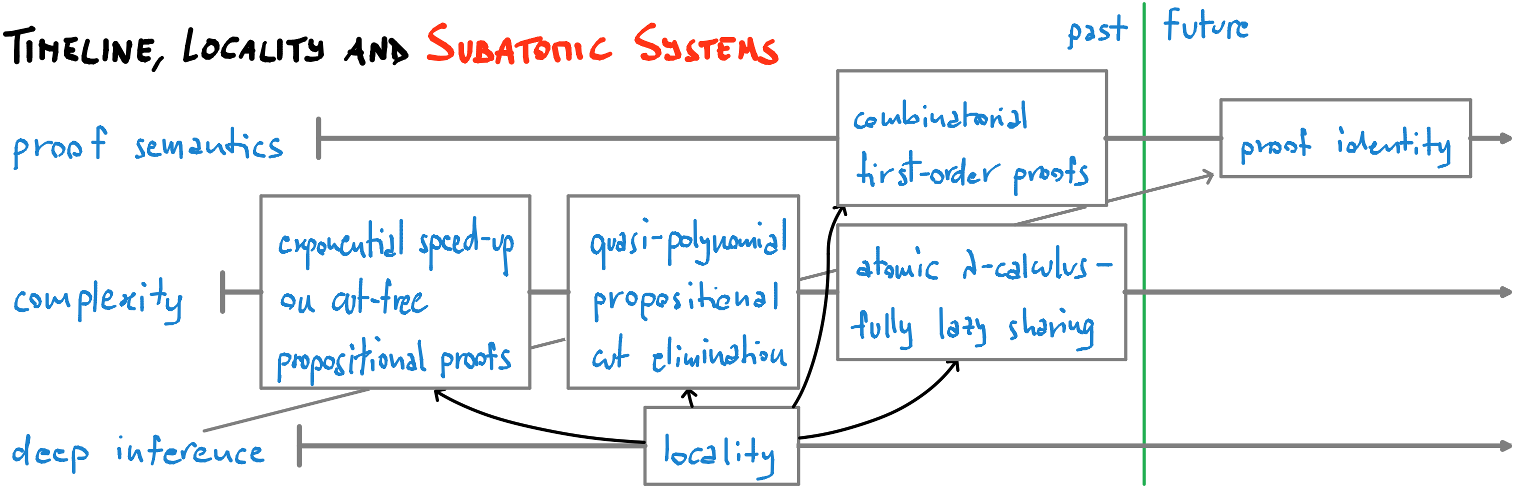
$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\alpha} = \hat{\alpha} = \alpha$

self-dual non-commutative 'atoms'

$\alpha, \beta \in \{\vee, \wedge, a, b, c, \dots\}$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



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$$\text{switch}\ \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \text{at. cut}\ \frac{a \wedge \bar{a}}{0}$$

$$\text{medial}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \text{at. contr.}\ \frac{a \vee a}{a}$$

instance of the shape

$$\frac{(0\, a\, 1) \wedge (1\, a\, 0)}{(0 \wedge 1)\, a\, (1 \wedge 0)}$$

superposition of truth values

shape

$$\frac{(A\, \dot{\beta}\, B)\, \alpha\, (C\, \dot{\beta}\, D)}{(A\, \dot{\alpha}\, C)\, \beta\, (B\, \dot{\alpha}\, D)}$$

saturation

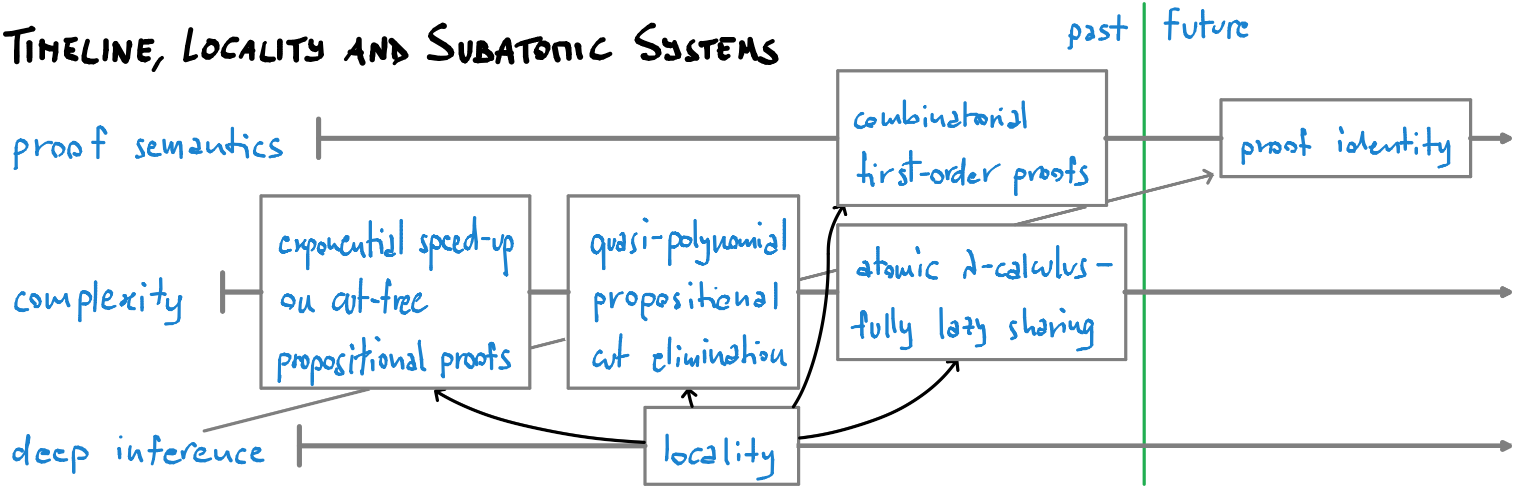
$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{\alpha} = \hat{\alpha} = \alpha$$

$$\alpha, \beta \in \{\vee, \wedge, a, b, c, \dots\}$$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



instances of the shape

switch $$\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

medial $$\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

at. cut $$\frac{a \wedge \bar{a}}{0} \longleftarrow \frac{(0 \, a \, 1) \wedge (1 \, a \, 0)}{(0 \wedge 1) \, a \, (1 \wedge 0)}$$

at. contr. $$\frac{a \vee a}{a}$$

instance of the shape

shape

$$\frac{(A \, \dot{\beta} \, B) \, \alpha \, (C \, \dot{\beta} \, D)}{(A \, \dot{\alpha} \, C) \, \beta \, (B \, \dot{\alpha} \, D)}$$

saturation

$$\check{\vee} = \check{\wedge} = \vee$$

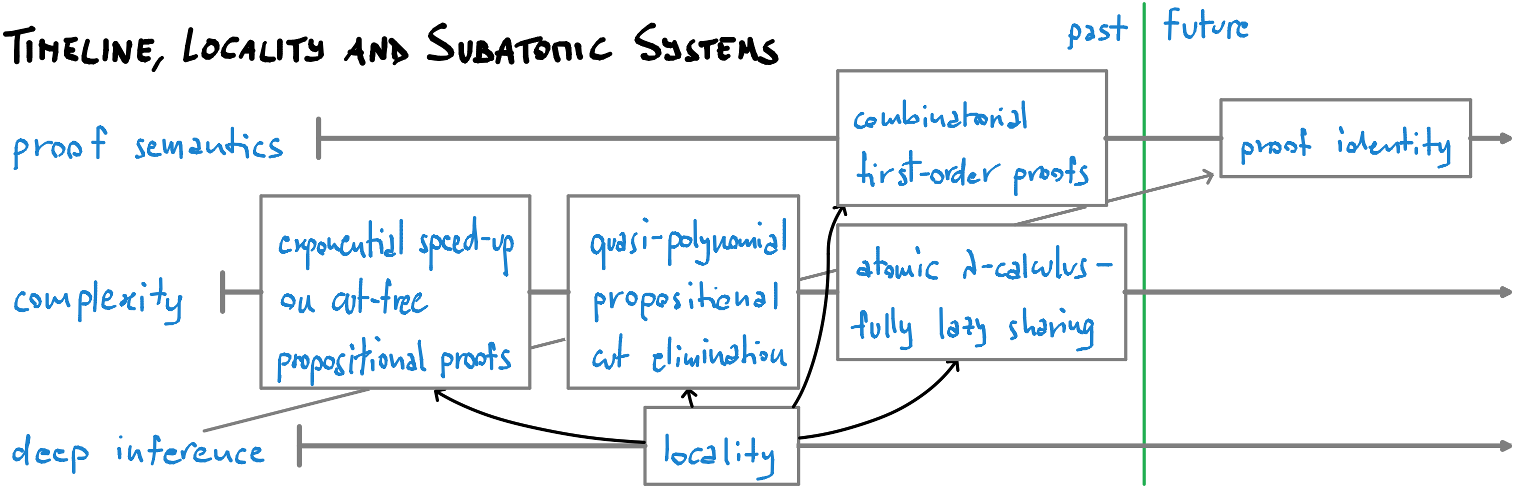
$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{\alpha} = \hat{\alpha} = \alpha$$

interpretation – CL

$$\alpha, \beta \in \{\vee, \wedge, a, b, c, \ldots\}$$

# Timeline, Locality and Subatomic Systems



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$$\text{switch}\ \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\text{medial}\ \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

instance of the shape

$$\text{at. cut}\ \frac{a \wedge \bar{a}}{0} \longmapsfrom \frac{(0\, a\, 1) \wedge (1\, a\, 0)}{(0 \wedge 1)\, a\, (1 \wedge 0)}$$

$$\text{at. contr.}\ \frac{a \vee a}{a} \longmapsfrom \frac{(0\, a\, 1) \vee (0\, a\, 1)}{(0 \vee 0)\, a\, (1 \vee 1)}$$

shape

$$\frac{(A\, \dot{\beta}\, B)\, \alpha\, (C\, \dot{\beta}\, D)}{(A\, \dot{\alpha}\, C)\, \beta\, (B\, \dot{\alpha}\, D)}$$

saturation

$$\check{\vee} = \check{\wedge} = \vee$$

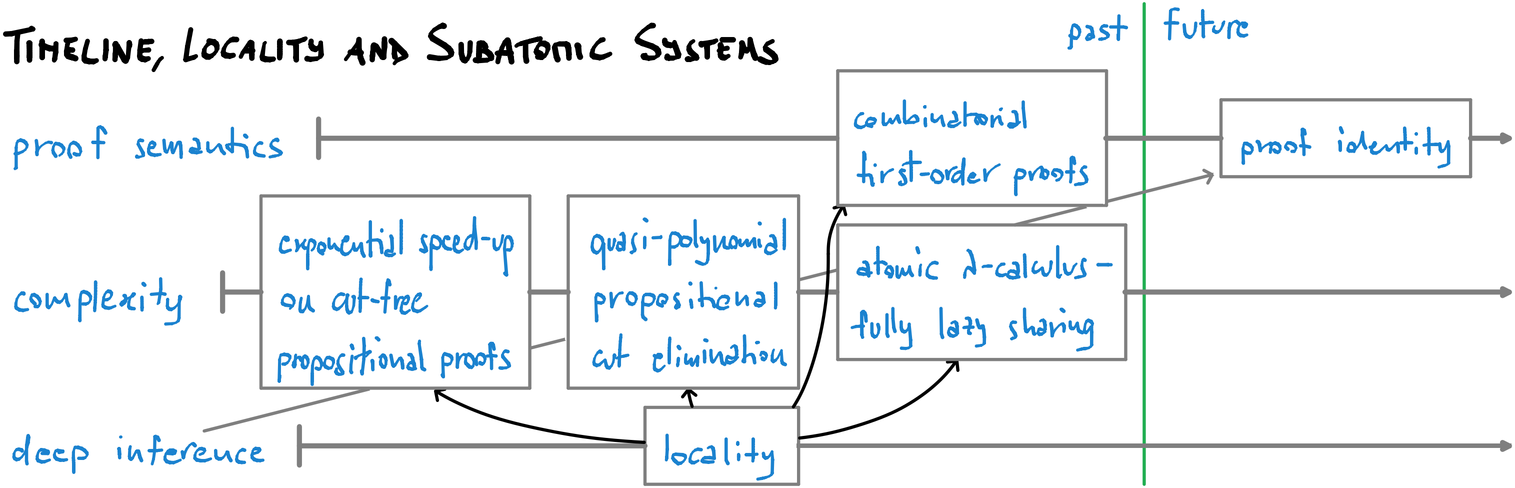
$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{\alpha} = \hat{\alpha} = \alpha$$

interpretation – CL

$$\alpha, \beta \in \{\vee, \wedge, a, b, c, \ldots\}$$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



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instances of the shape

$$\text{switch}\ \frac{(A ⅋ B) \otimes (C \otimes D)}{(A \otimes C) ⅋ (B \otimes D)}$$

$$\text{medial}\ \frac{(A \otimes B) ⅋ (C \otimes D)}{(A ⅋ C) \otimes (B ⅋ D)}$$

$$\text{at. cut}\ \frac{a \otimes \bar{a}}{0} \longleftarrow \frac{(\bot\ a\ 1) \otimes (1\ a\ \bot)}{(\bot \otimes 1)\ a\ (1 \otimes \bot)}$$

$$\text{at. contr.}\ \frac{a ⅋ a}{a} \not\longleftarrow \frac{(\bot\ a\ 1) ⅋ (\bot\ a\ 1)}{(\bot ⅋ \bot)\ a\ (1 ⅋ 1)}$$

instance of the shape

shape

$$\frac{(A\ \dot{\beta}\ B)\ \alpha\ (C\ \dot{\beta}\ D)}{(A\ \dot{\alpha}\ C)\ \beta\ (B\ \dot{\alpha}\ D)}$$

saturation

$$\check{⅋} = \check{\otimes} = ⅋$$

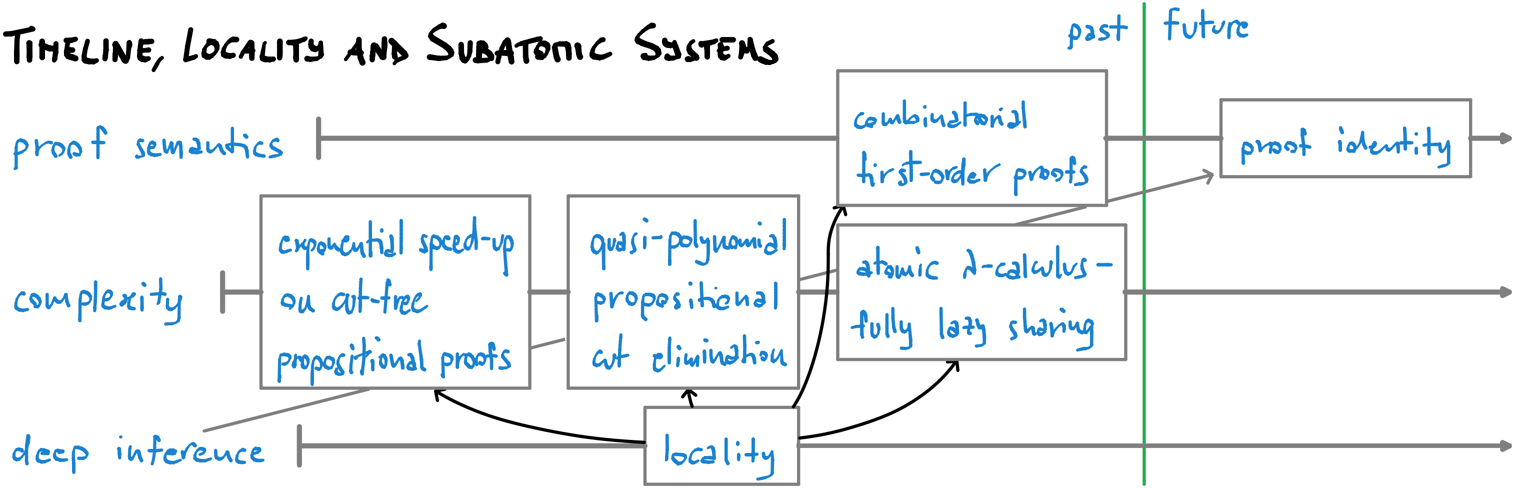
$$\hat{⅋} = \hat{\otimes} = \otimes$$

$$\check{\alpha} = \hat{\alpha} = \alpha$$

interpretation – LL

$$\alpha, \beta \in \{⅋, \otimes, \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



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locality

1 study subatomic normalisation

2 interpret it for different logics (CL, LL, BV, ...)

[Alex Tubella, Guglielmi, ACM ToCL, 2018]

standard deep inference

$$\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \qquad \frac{a \wedge \bar{a}}{0}$$

$$\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \frac{a \vee a}{a}$$

←

subatomic proof systems

shape

$$\frac{(A \dot{\beta} B)\ \alpha\ (C \dot{\beta} D)}{(A \dot{\alpha} C)\ \beta\ (B \dot{\alpha} D)}$$

+ unit equations

saturation

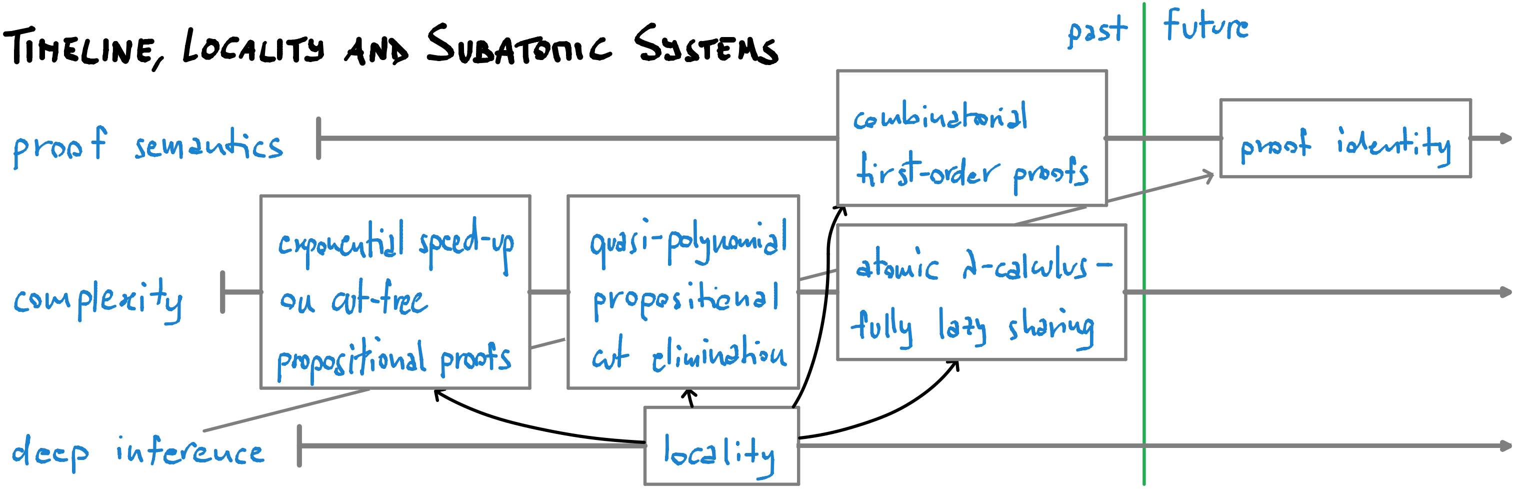
$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{\alpha} = \hat{\alpha} = \alpha$$

$$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS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the shape works for unary connectives

$$\frac{(\dot{\beta} A)\ \alpha\ (\dot{\beta} B)}{\beta (A \dot{\alpha} B)}$$

$$\frac{\alpha (A \dot{\beta} B)}{(\dot{\alpha} A)\ \beta\ (\dot{\alpha} B)}$$

shape

$$\frac{(A \dot{\beta} B)\ \alpha\ (C \dot{\beta} D)}{(A \dot{\alpha} C)\ \beta\ (B \dot{\alpha} D)}$$

+ unit equations

saturation

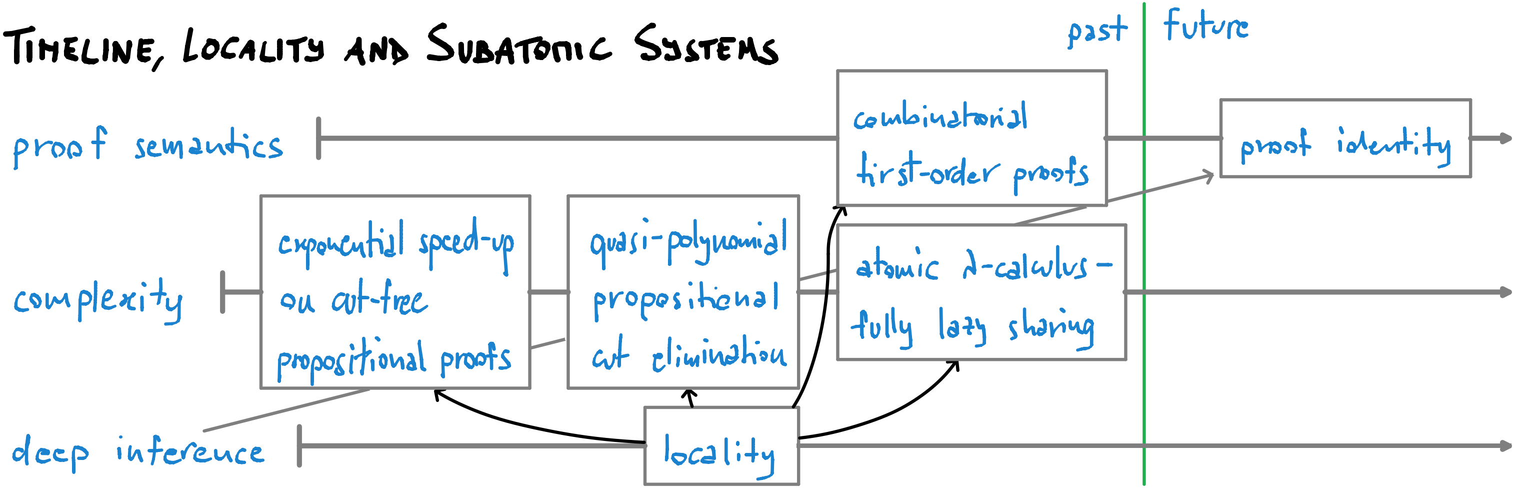
$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{\alpha} = \hat{\alpha} = \alpha$$

$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



the shape works for unary connectives

$$\frac{\Diamond A \wedge \Box B}{\Diamond(A \wedge B)} \quad \leftarrow \quad \frac{(\dot{\beta} A)\, \alpha\, (\dot{\beta} B)}{\beta(A\, \dot{\alpha}\, B)}$$

$$\frac{\Box(A \vee B)}{\Box A \vee \Diamond B} \quad \leftarrow \quad \frac{\alpha(A\, \dot{\beta}\, B)}{(\dot{\alpha} A)\, \beta\, (\dot{\alpha} B)}$$

shape

$$\frac{(A\, \dot{\beta}\, B)\, \alpha\, (C\, \dot{\beta}\, D)}{(A\, \dot{\alpha}\, C)\, \beta\, (B\, \dot{\alpha}\, D)}$$

+ unit equations

saturation

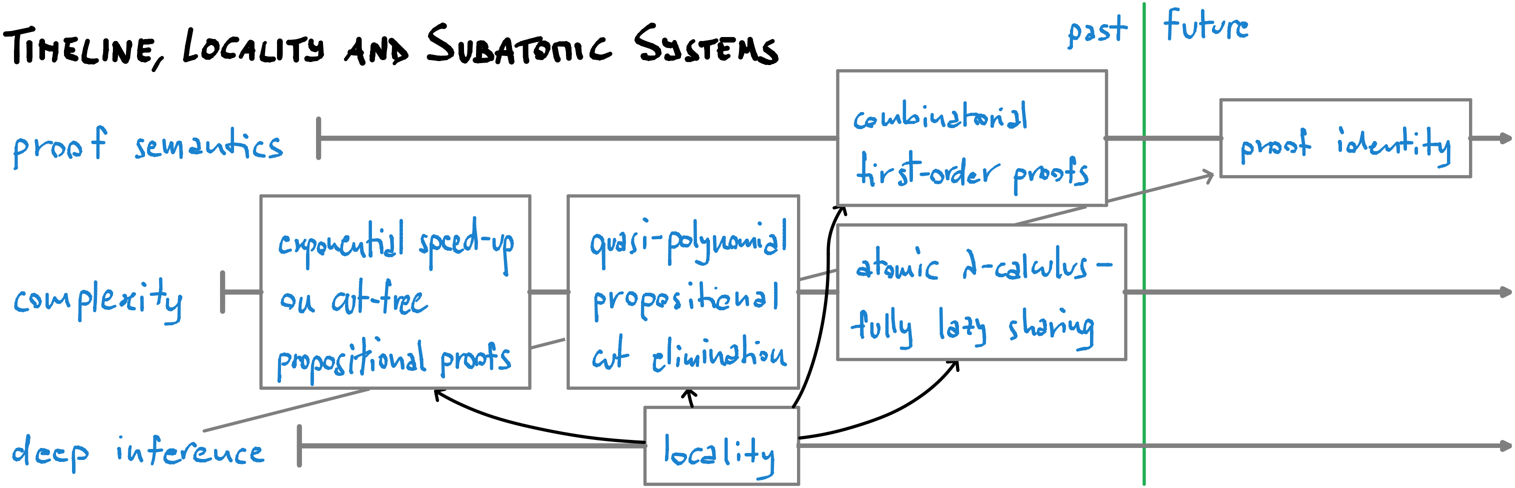
$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\alpha} = \hat{\alpha} = \alpha$

$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



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the shape works for unary connectives

$$\frac{\exists x. A \wedge \forall x. B}{\exists x.(A \wedge B)} \quad \leftarrow \quad \frac{(\dot{\beta} A)\ \alpha\ (\dot{\beta} B)}{\beta (A\ \dot{\alpha}\ B)}$$

$$\frac{\forall x.(A \vee B)}{\forall x. A \vee \exists x. B} \quad \leftarrow \quad \frac{\alpha (A\ \dot{\beta}\ B)}{(\dot{\alpha} A)\ \beta\ (\dot{\alpha} B)}$$

shape

$$\frac{(A\ \dot{\beta}\ B)\ \alpha\ (C\ \dot{\beta}\ D)}{(A\ \dot{\alpha}\ C)\ \beta\ (B\ \dot{\alpha}\ D)}$$

+ unit equations

saturation

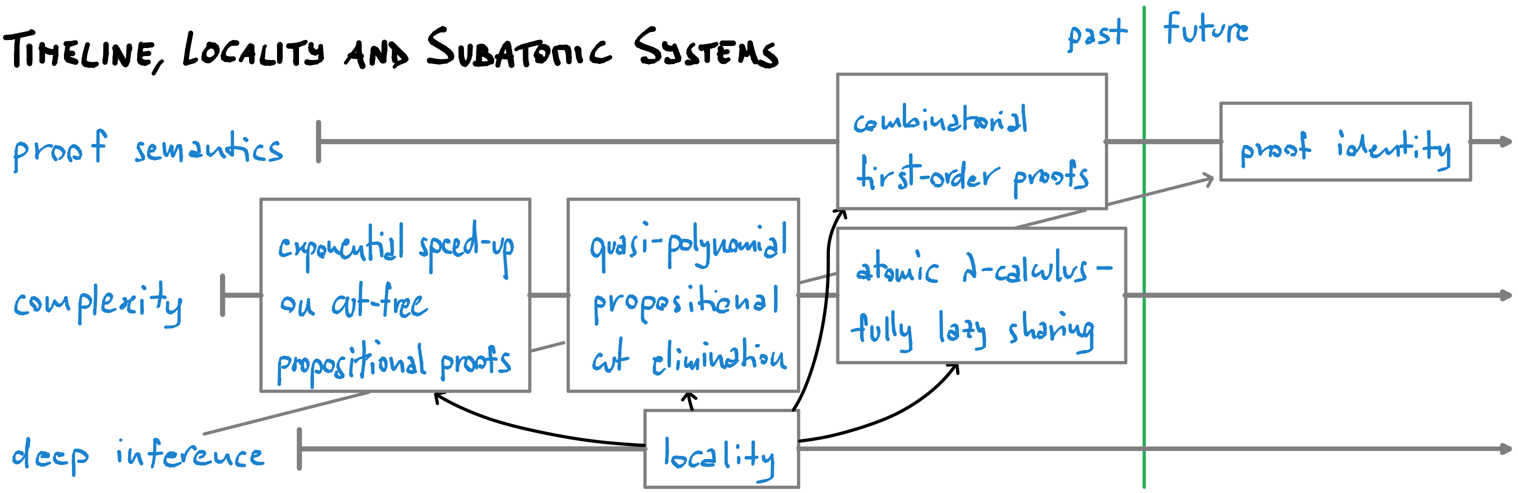
$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\boldsymbol{\alpha}} = \hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$

$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



the shape works for unary connectives

$$\frac{\exists x.\forall y.A}{\forall y.\exists x.A} \leftarrow \frac{\alpha(\dot{\beta} A)}{\beta(\dot{\alpha} A)}$$

$$\frac{\exists x.A \wedge \forall x.B}{\exists x.(A \wedge B)} \leftarrow \frac{(\dot{\beta} A)\, \alpha\, (\dot{\beta} B)}{\beta(A\, \dot{\alpha}\, B)}$$

$$\frac{\forall x.(A \vee B)}{\forall x.A \vee \exists x.B} \leftarrow \frac{\alpha(A\, \dot{\beta}\, B)}{(\dot{\alpha} A)\, \beta\, (\dot{\alpha} B)}$$

shape

$$\frac{(A\, \dot{\beta}\, B)\, \alpha\, (C\, \dot{\beta}\, D)}{(A\, \dot{\alpha}\, C)\, \beta\, (B\, \dot{\alpha}\, D)}$$

+ unit equations

saturation

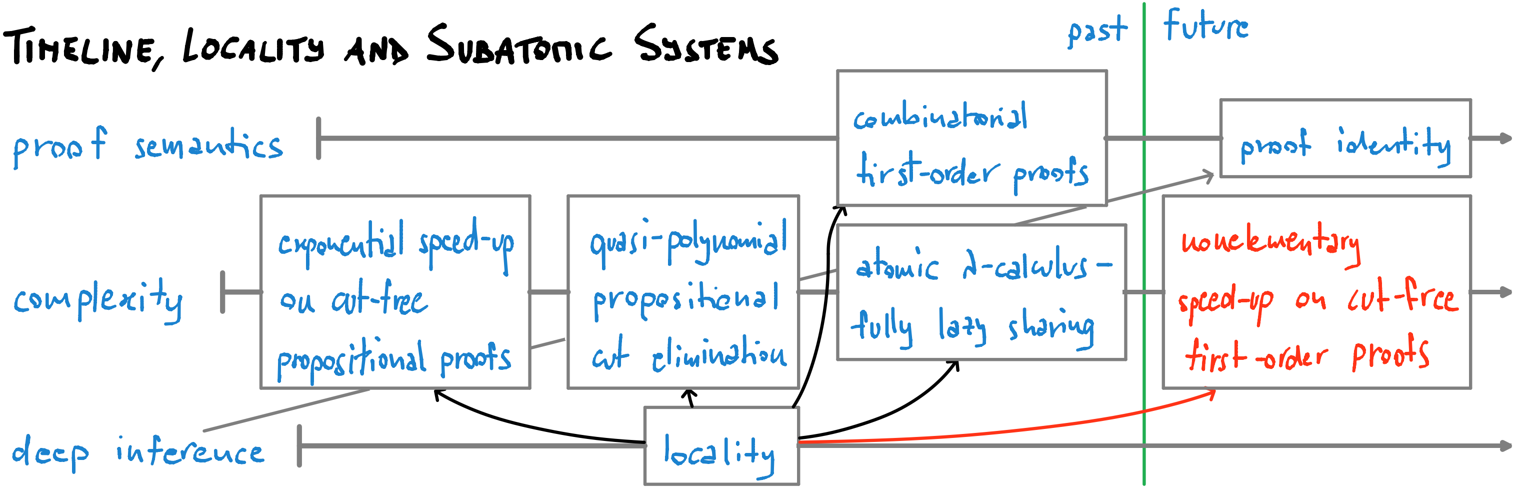
$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\alpha} = \hat{\alpha} = \alpha$

$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$

# TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



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locality

the shape works for unary connectives

$$\frac{\exists x.\forall y.A}{\forall y.\exists x.A} \leftarrow \frac{\alpha(\dot{\beta} A)}{\beta(\dot{\alpha} A)}$$

$$\frac{\exists x.A \wedge \forall x.B}{\exists x.(A \wedge B)} \leftarrow \frac{(\dot{\beta} A)\,\alpha\,(\dot{\beta} B)}{\beta(A\,\dot{\alpha}\,B)}$$

$$\frac{\forall x.(A \vee B)}{\forall x.A \vee \exists x.B} \leftarrow \frac{\alpha(A\,\dot{\beta}\,B)}{(\dot{\alpha} A)\,\beta\,(\dot{\alpha} B)}$$

quantifier shifts!

shape

$$\frac{(A\,\dot{\beta}\,B)\,\alpha\,(C\,\dot{\beta}\,D)}{(A\,\dot{\alpha}\,C)\,\beta\,(B\,\dot{\alpha}\,D)}$$

+ unit equations

saturation

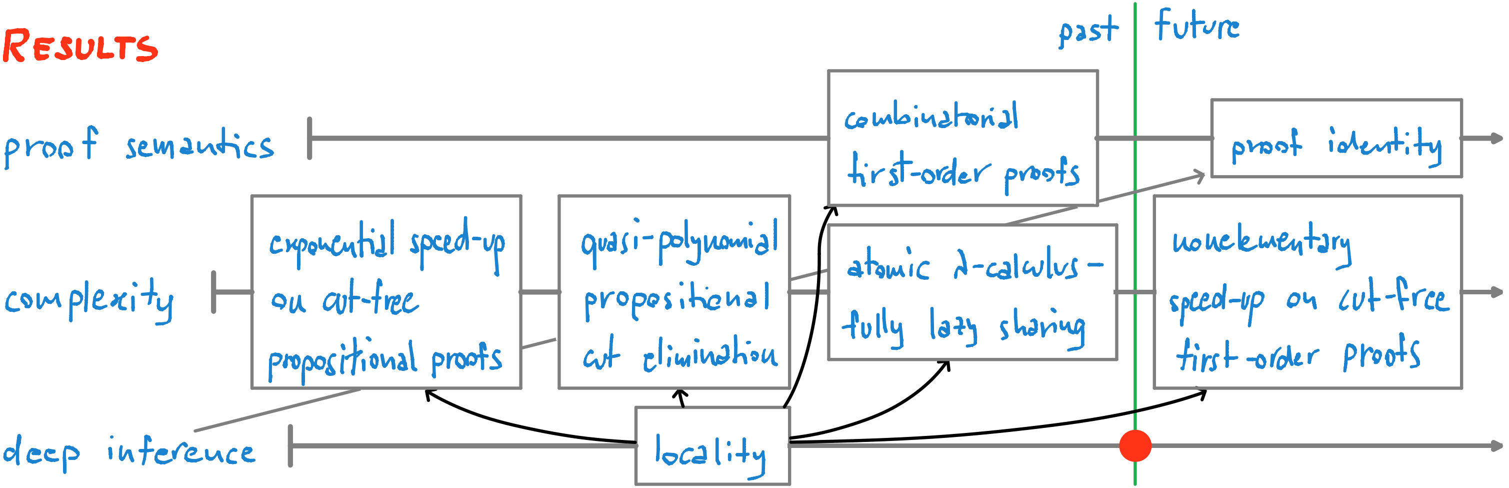
$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\alpha} = \hat{\alpha} = \alpha$

$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \dots\}$

# RESULTS



1

Take all instances of the shape. You get:

- a proof system for propositional classical logic with decision trees

A if **a** false and **b** false

B if **a** true and **b** false

...

invertible rule

$$\frac{(A\ a\ B)\ b\ (C\ a\ D)}{(A\ b\ C)\ a\ (B\ b\ D)}$$

shape

$$\frac{(A\ \dot{\beta}\ B)\ \alpha\ (C\ \dot{\beta}\ D)}{(A\ \dot{\alpha}\ C)\ \beta\ (B\ \dot{\alpha}\ D)}$$

\+ unit equations

saturation

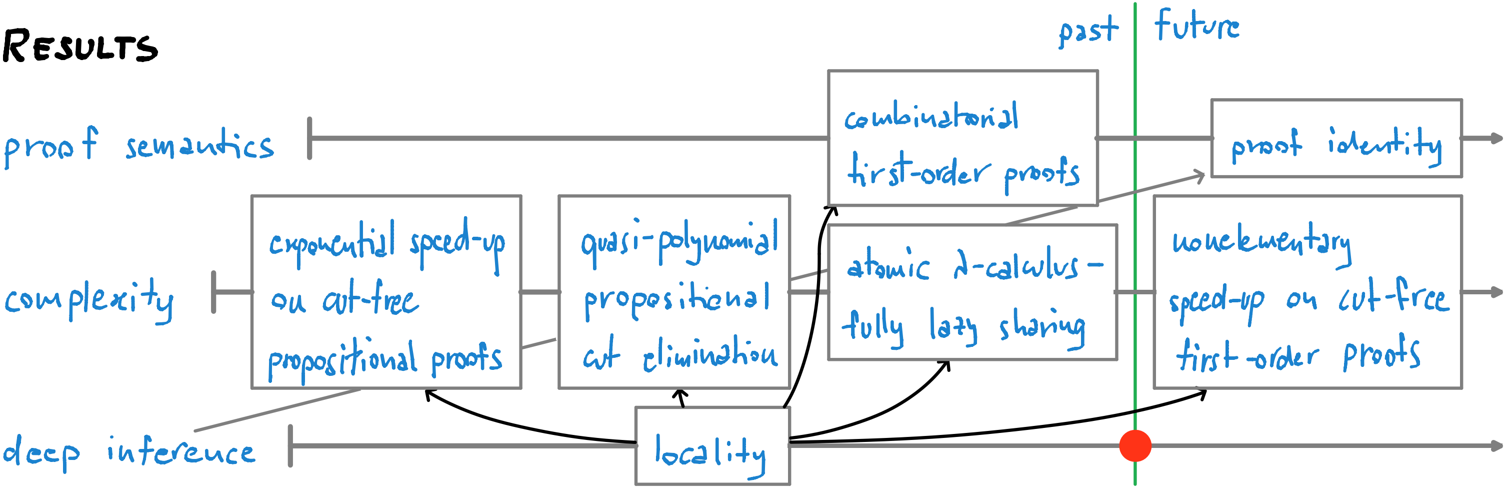
$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\alpha} = \hat{\alpha} = \alpha$

$\alpha, \beta \in \{\vee, \wedge, a, b, c, \dots\}$

# RESULTS



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locality

1

Take all instances of the shape. You get:

- a proof system for propositional classical logic with decision trees;
- a simple cut-elimination procedure based on projections

shape

$$\frac{(A \mathbin{\dot{\beta}} B) \mathbin{\alpha} (C \mathbin{\dot{\beta}} D)}{(A \mathbin{\dot{\alpha}} C) \mathbin{\beta} (B \mathbin{\dot{\alpha}} D)}$$

+ unit equations

saturation

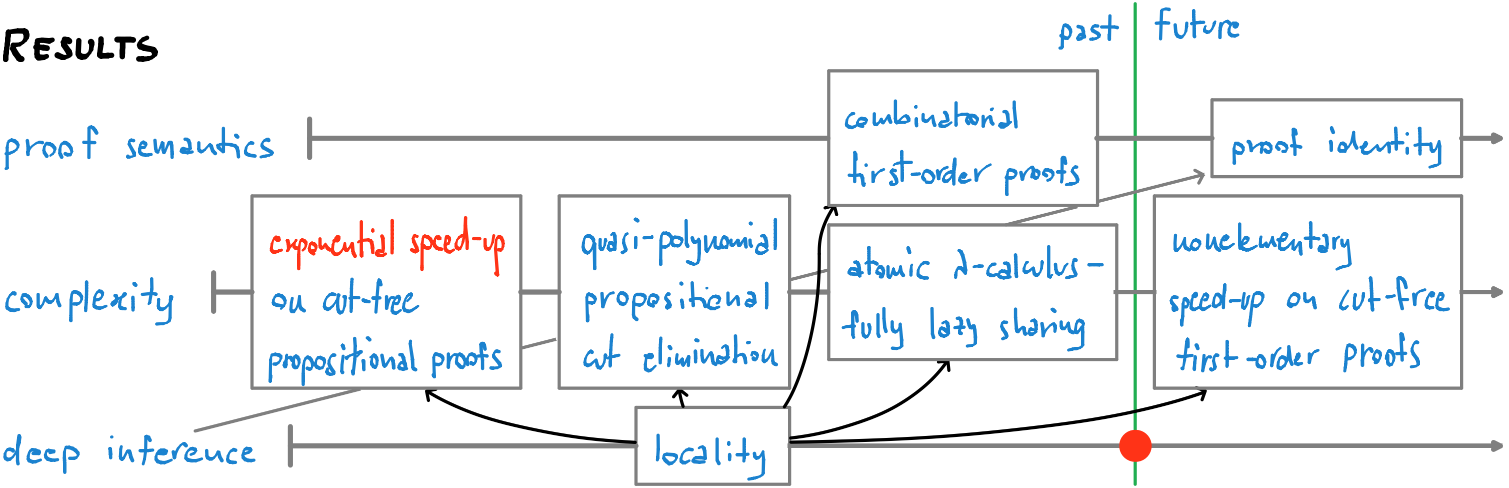
$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\alpha} = \hat{\alpha} = \alpha$

$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \dots\}$

# RESULTS



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1

Take all instances of the shape. You get:

- a proof system for propositional classical logic with decision trees;
- a simple cut-elimination procedure based on projections;
- projections build polynomial-size proofs of Statman tautologies.

[C. Barrett, Guglielmi, ACM ToCL, to appear]

shape

$$\frac{(A \mathbin{\dot\beta} B) \mathbin{\alpha} (C \mathbin{\dot\beta} D)}{(A \mathbin{\dot\alpha} C) \mathbin{\beta} (B \mathbin{\dot\alpha} D)}$$

+ unit equations

saturation

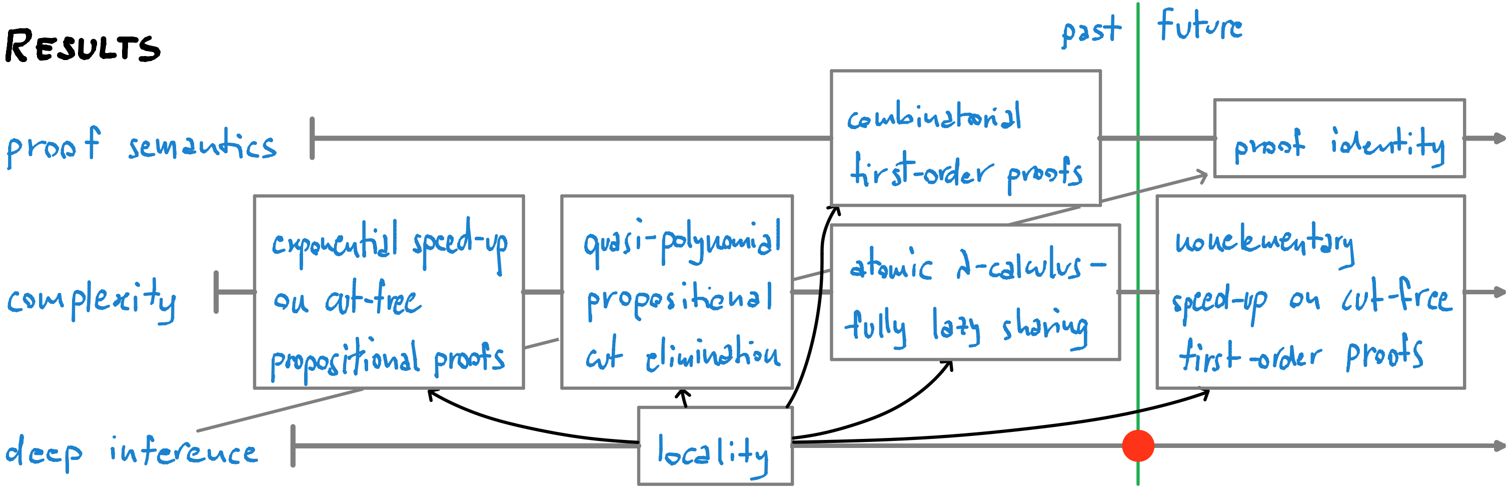
$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\alpha} = \hat{\alpha} = \alpha$

$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$

# RESULTS



2 Take the unit equations

$$\frac{A}{A \vee 0} \qquad \frac{A \vee 0}{A} \qquad \frac{A}{A \wedge 1} \qquad \frac{A \wedge 1}{A}$$

$$\frac{0}{0 \wedge 0} \qquad \frac{0}{0 \mathbf{a} 0} \qquad \frac{0 \mathbf{a} 0}{0} \qquad \frac{1 \vee 1}{1} \qquad \frac{1 \mathbf{a} 1}{1} \qquad \frac{1}{1 \mathbf{a} 1}$$

shape

$$\frac{(A \dot{\beta} B) \,\alpha\, (C \dot{\beta} D)}{(A \dot{\alpha} C) \,\beta\, (B \dot{\alpha} D)}$$

+ unit equations

saturation

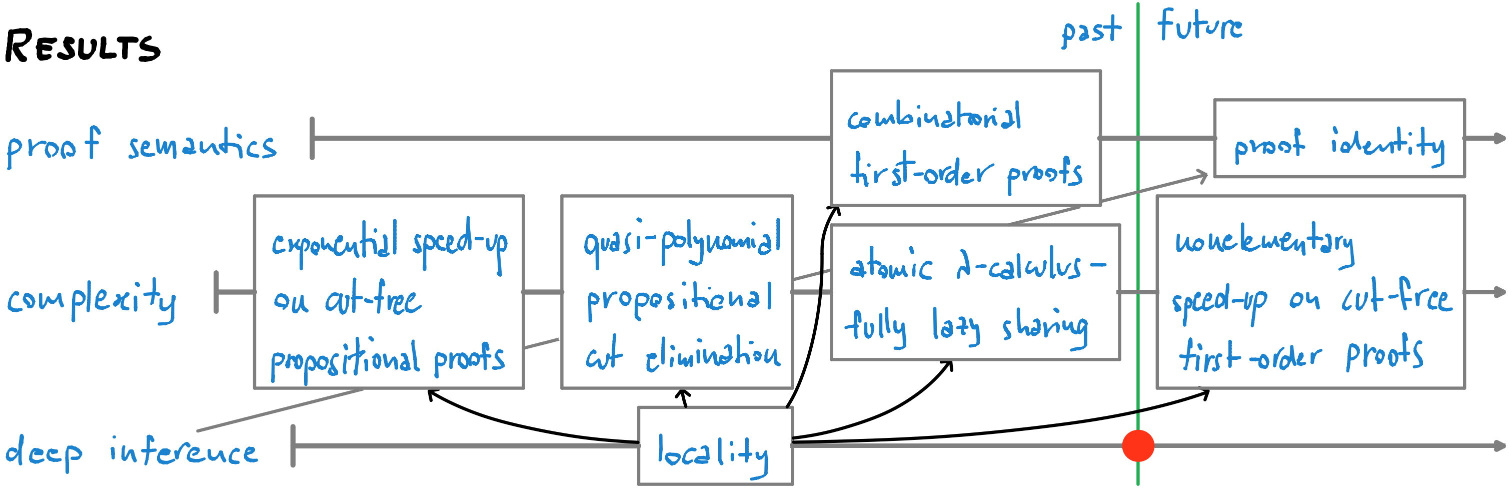
$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{\boldsymbol{\alpha}} = \hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$$

$$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$$

# RESULTS



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2

Take the unit equations:

- they are admissible

$$\frac{A}{A \vee 0} \qquad \frac{A \vee 0}{A} \qquad \frac{A}{A \wedge 1} \qquad \frac{A \wedge 1}{A}$$

$$\frac{0}{0 \wedge 0} \qquad \frac{0}{0 \,\mathbf{a}\, 0} \qquad \frac{0 \,\mathbf{a}\, 0}{0} \qquad \frac{1 \vee 1}{1} \qquad \frac{1 \,\mathbf{a}\, 1}{1} \qquad \frac{1}{1 \,\mathbf{a}\, 1}$$

shape

$$\frac{(A \,\dot{\beta}\, B) \,\alpha\, (C \,\dot{\beta}\, D)}{(A \,\dot{\alpha}\, C) \,\beta\, (B \,\dot{\alpha}\, D)}$$

~~+ unit equations~~

saturation

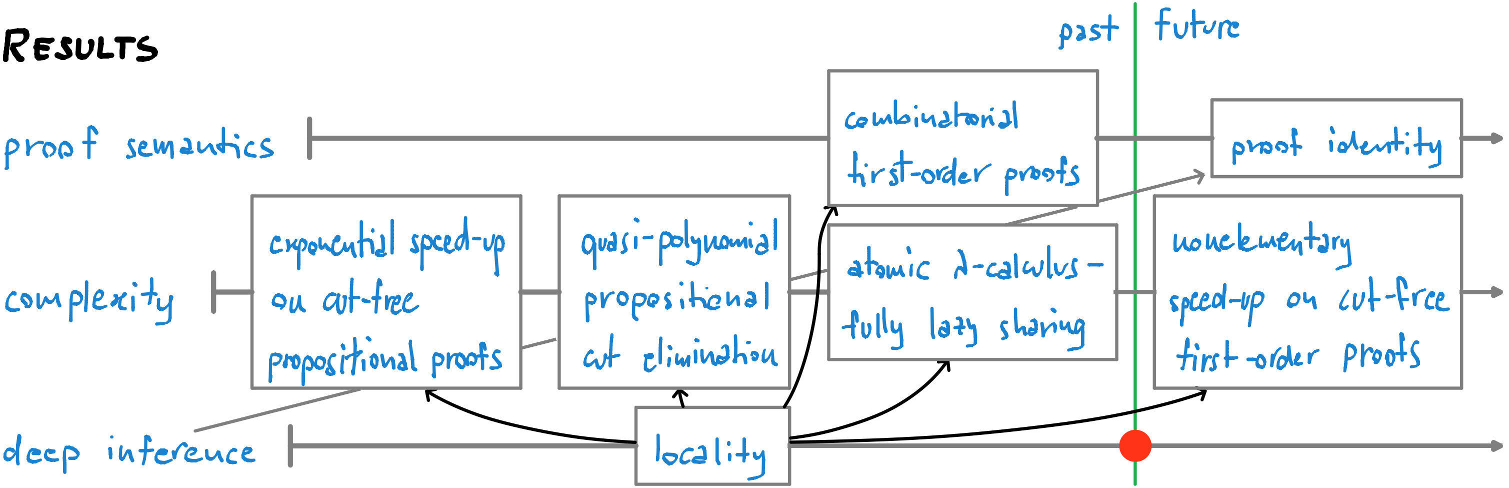
$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{\alpha} = \hat{\alpha} = \alpha$$

$$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$$

# RESULTS


past

future

proof semantics

complexity

deep inference

combinatorial first-order proofs

proof identity

exponential speed-up on cut-free propositional proofs

quasi-polynomial propositional cut elimination

atomic λ-calculus – fully lazy sharing

nonelementary speed-up on cut-free first-order proofs

locality

2

Take the unit equations:

- they are admissible,
- for several logics.

shape

$$\frac{(A \dot{\beta} B)\ \alpha\ (C \dot{\beta} D)}{(A \dot{\alpha} C)\ \beta\ (B \dot{\alpha} D)}$$

saturation

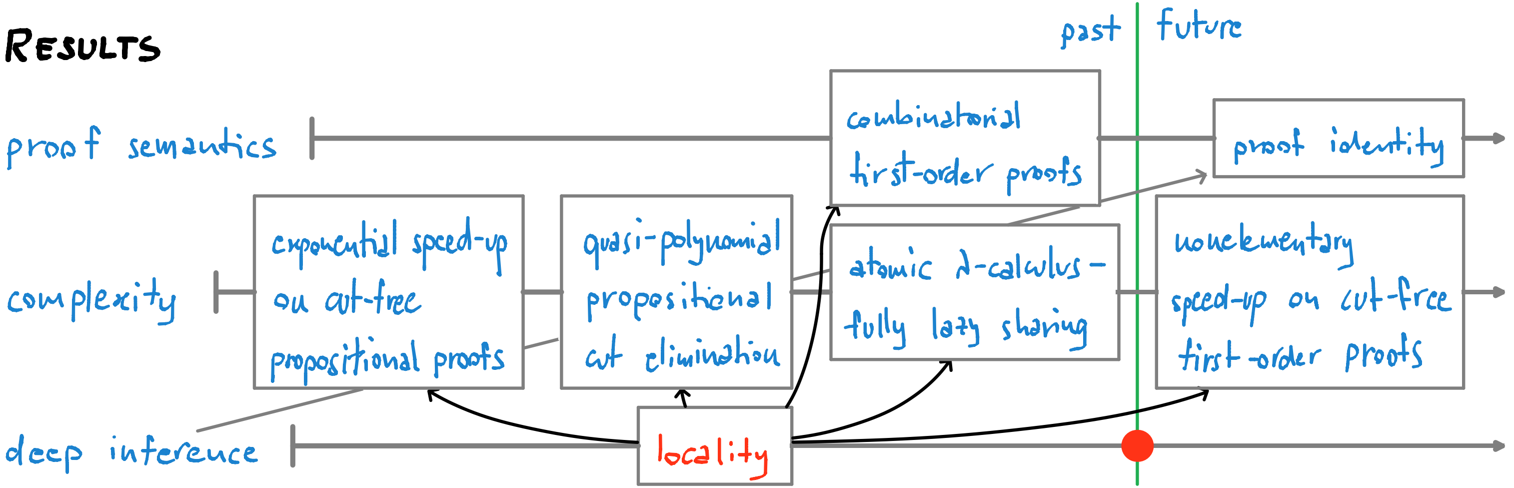
~~$\check{\vee} = \check{\wedge} = \vee$~~

~~$\hat{\vee} = \hat{\wedge} = \wedge$~~

~~$\check{\alpha} = \hat{\alpha} = \alpha$~~

~~$\alpha, \beta \in \{\vee, \wedge, a, b, c, \ldots\}$~~

# RESULTS



past | future

proof semantics

complexity

deep inference

combinatorial first-order proofs

proof identity

exponential speed-up on cut-free propositional proofs

quasi-polynomial propositional cut elimination

atomic λ-calculus – fully lazy sharing

nonelementary speed-up on cut-free first-order proofs

locality

2

Take the unit equations:

- they are admissible,
- for several logics.

→ total linearity

[V. Barrett, Guglielmi, TLLA 2021, in preparation]

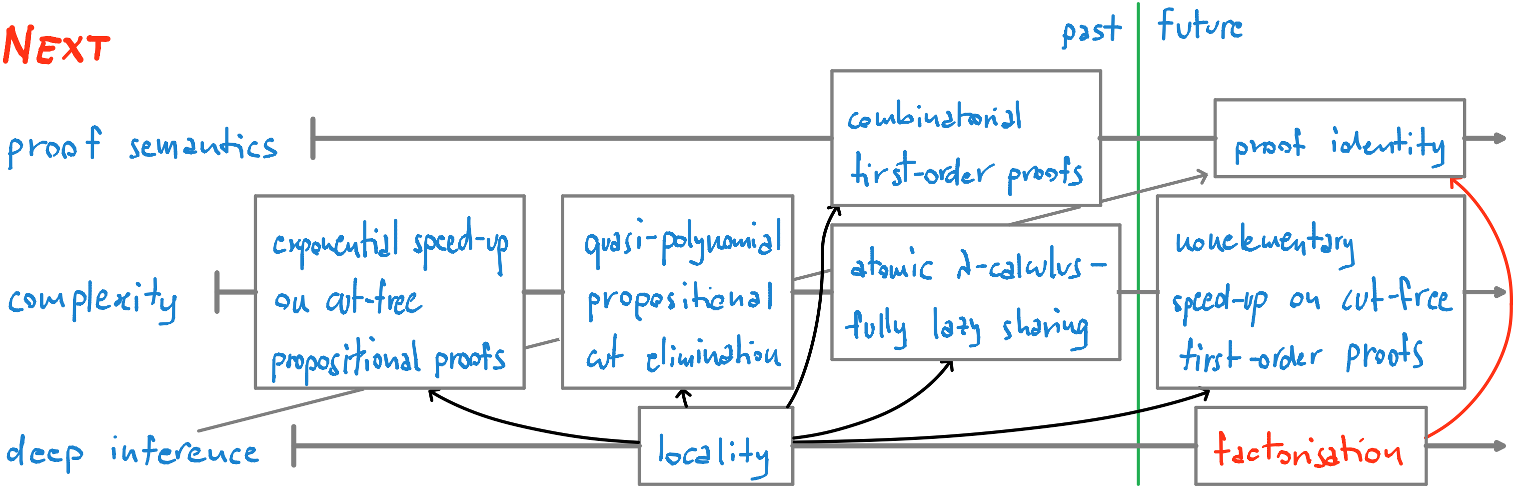
shape

$$\frac{(A \,\beta\, B) \,\alpha\, (C \,\beta\, D)}{(A \,\alpha\, C) \,\beta\, (B \,\alpha\, D)}$$

saturation

logic dependent

# NEXT



Take the unit equations:

- they are admissible,
- for several logics.

→ total linearity

→ proof substitution and factorisation

shape

$$\frac{(A \dot{\beta} B) \alpha (C \dot{\beta} D)}{(A \dot{\alpha} C) \beta (B \dot{\alpha} D)}$$

saturation

logic dependent

# ADMISSIBILITY OF UNIT EQUATIONS

$$\frac{(A \dot{\beta} B)\ \alpha\ (C \dot{\beta} D)}{(A \dot{\alpha} C)\ \beta\ (B \dot{\alpha} D)}$$

$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\boldsymbol{\alpha}} = \hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$

$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \dots\}$

$$\frac{A}{A \vee 0} \qquad \frac{A \vee 0}{A} \qquad \frac{A}{A \wedge 1} \qquad \frac{A \wedge 1}{A}$$

$$\frac{0}{0 \wedge 0} \qquad \frac{0}{0 \mathbf{a} 0} \qquad \frac{0 \mathbf{a} 0}{0}$$

$$\frac{1 \vee 1}{1} \qquad \frac{1 \mathbf{a} 1}{1} \qquad \frac{1}{1 \mathbf{a} 1}$$

propositional classical logic
(with decision trees)

# ADMISSIBILITY OF UNIT EQUATIONS

**Lemma** (1) are admissible.

**Proof** Move steps up and down. Steps vanish at top and bottom. Interpretation does not change. Example:

$$\begin{array}{c} 0\,a\,0 \\ \hline 0 \\ \hline 0\wedge 0 \end{array} \quad \longrightarrow \quad \begin{array}{c} \dfrac{0}{0\wedge 0}\; a \; \dfrac{0}{0\wedge 0} \\ \hline \dfrac{0\,a\,0}{0} \;\wedge\; \dfrac{0\,a\,0}{0} \end{array}$$

propositional classical logic
(with decision trees)

$$\frac{(A \overset{\bullet}{\beta} B)\; \alpha \;(C \overset{\bullet}{\beta} D)}{(A \overset{\bullet}{\alpha} C)\; \beta \;(B \overset{\bullet}{\alpha} D)}$$

$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\alpha} = \hat{\alpha} = \alpha$

$\alpha, \beta \in \{\vee, \wedge, a, b, c, \dots\}$

$$\frac{A}{A \vee 0} \qquad \frac{A \vee 0}{A} \qquad \frac{A}{A \wedge 1} \qquad \frac{A \wedge 1}{A}$$

(1)

$$\frac{0}{0 \wedge 0} \qquad \frac{0}{0\,a\,0} \qquad \frac{0\,a\,0}{0}$$

$$\frac{1 \vee 1}{1} \qquad \frac{1\,a\,1}{1} \qquad \frac{1}{1\,a\,1}$$

# ADMISSIBILITY OF UNIT EQUATIONS

**Lemma** (1) are admissible.

**Lemma** (Eversion) Given $A$ and $B$, there exist

$$\begin{array}{c}[B^i \Rightarrow x_i]_{\check{\underline{A}}}\, \check{A} \\ \| \\ [\check{A}^j \Rightarrow y_j]_{\underline{B}}\, B\end{array}$$

where $B^i$'s (resp. $\check{A}^j$'s) are renamings of $B$ (resp. $\check{A}$) and $\bigcup_i \underline{B}^i = \bigcup_j \underline{\check{A}}^j$.

$$\frac{(A \dot{\beta} B)\ \alpha\ (C \dot{\beta} D)}{(A \dot{\alpha} C)\ \beta\ (B \dot{\alpha} D)}$$

$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\boldsymbol{\alpha}} = \hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$

$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \dots\}$

$$\frac{A}{A \vee 0} \qquad \frac{A \vee 0}{A} \qquad \frac{A}{A \wedge 1} \qquad \frac{A \wedge 1}{A}$$

(1)

$$\frac{0}{0 \wedge 0} \qquad \frac{0}{0\, \mathbf{a}\, 0} \qquad \frac{0\, \mathbf{a}\, 0}{0}$$

$$\frac{1 \vee 1}{1} \qquad \frac{1\, \mathbf{a}\, 1}{1} \qquad \frac{1}{1\, \mathbf{a}\, 1}$$

# ADMISSIBILITY OF UNIT EQUATIONS

**Lemma** (1) are admissible.

**Lemma** (Eversion) Given $A$ and $B$, there exist

$$[B^i \Rightarrow x_i]_{\underline{\check{A}}}\, \check{A}$$

$A$ connectives saturated down

$$\|$$

substitute into variables of $B$

$$[\check{A}^j \Rightarrow y_j]_{\underline{B}}\, B$$

where $B^i$'s (resp. $\check{A}^j$'s) are renamings of $B$ (resp. $\check{A}$) and $\bigcup_i \underline{B}^i = \bigcup_j \underline{\check{A}}^j$.

**Proof** Use the shape in a straightforward induction.

$$\frac{(A \dot{\beta} B)\ \alpha\ (C \dot{\beta} D)}{(A \dot{\alpha} C)\ \beta\ (B \dot{\alpha} D)}$$

$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\alpha} = \hat{\alpha} = \alpha$

$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \dots\}$

$$\frac{A}{A \vee 0} \qquad \frac{A \vee 0}{A} \qquad \frac{A}{A \wedge 1} \qquad \frac{A \wedge 1}{A}$$

(1)

$$\frac{0}{0 \wedge 0} \qquad \frac{0}{0\, \mathbf{a}\, 0} \qquad \frac{0\, \mathbf{a}\, 0}{0}$$

$$\frac{1 \vee 1}{1} \qquad \frac{1\, \mathbf{a}\, 1}{1} \qquad \frac{1}{1\, \mathbf{a}\, 1}$$

# ADMISSIBILITY OF UNIT EQUATIONS

**Lemma** (1) are admissible.

**Lemma** (Eversion) Given $A$ and $B$, there exist

$$\begin{array}{c}[B^i \Rightarrow x_i]_{\underline{\check{A}}}\, \check{A} \\ \| \\ [\check{A}^j \Rightarrow y_j]_{\underline{B}}\, B\end{array}.$$

where $B^i$'s (resp. $\check{A}^j$'s) are renamings of $B$ (resp. $\check{A}$) and $\bigcup_i \underline{B^i} = \bigcup_j \underline{\check{A}^j}$.

**Theorem** (2) are admissible.

$$\frac{(A \,\dot{\beta}\, B) \;\alpha\; (C \,\dot{\beta}\, D)}{(A \,\dot{\alpha}\, C) \;\beta\; (B \,\dot{\alpha}\, D)}$$

$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{\mathbf{a}} = \hat{\mathbf{a}} = \mathbf{a}$$

(2) $\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$

$$\frac{A}{A \vee 0} \qquad \frac{A \vee 0}{A} \qquad \frac{A}{A \wedge 1} \qquad \frac{A \wedge 1}{A}$$

(1)

$$\frac{0}{0 \wedge 0} \qquad \frac{0}{0 \,\mathbf{a}\, 0} \qquad \frac{0 \,\mathbf{a}\, 0}{0}$$

$$\frac{1 \vee 1}{1} \qquad \frac{1 \,\mathbf{a}\, 1}{1} \qquad \frac{1}{1 \,\mathbf{a}\, 1}$$

# ADMISSIBILITY OF UNIT EQUATIONS

**Lemma** (1) are admissible.

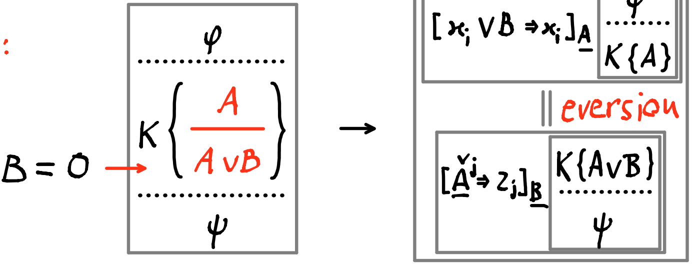
**Lemma** (Eversion) Given $A$ and $B$, there exist

$$\begin{array}{c}[B^i \Rightarrow x_i]_{\underline{\check{A}}}\, \check{A} \\ \| \\ [\check{A}^j \Rightarrow y_j]_{\underline{B}}\, B\end{array}$$

where $B^i$'s (resp. $\check{A}^j$s) are renamings of $B$ (resp. $\check{A}$) and $\bigcup_i \underline{B^i} = \bigcup_j \underline{\check{A}^j}$.

**Theorem** (2) are admissible.

**Proof** Main idea:



$$\frac{(A \dot{\beta} B)\ \alpha\ (C \dot{\beta} D)}{(A \dot{\alpha} C)\ \beta\ (B \dot{\alpha} D)}$$

$\check{\vee} = \check{\wedge} = \vee$

$\hat{\vee} = \hat{\wedge} = \wedge$

$\check{\alpha} = \hat{\alpha} = \alpha$

(2) $\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots\}$

$$\frac{A}{A \vee 0} \qquad \frac{A \vee 0}{A} \qquad \frac{A}{A \wedge 1} \qquad \frac{A \wedge 1}{A}$$

(1)

$$\frac{0}{0 \wedge 0} \qquad \frac{0}{0\, \mathbf{a}\, 0} \qquad \frac{0\, \mathbf{a}\, 0}{0}$$

$$\frac{1 \vee 1}{1} \qquad \frac{1\, \mathbf{a}\, 1}{1} \qquad \frac{1}{1\, \mathbf{a}\, 1}$$

# TOTAL LINEARITY

$$\frac{(A \,\dot{\beta}\, B) \,\alpha\, (C \,\dot{\beta}\, D)}{(A \,\dot{\alpha}\, C) \,\beta\, (B \,\dot{\alpha}\, D)}$$

$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{\boldsymbol{\alpha}} = \hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$$

$$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \dots\}$$

# TOTAL LINEARITY

Simple, natural proof theory.

$$\frac{(A \,\dot{\beta}\, B) \;\alpha\; (C \,\dot{\beta}\, D)}{(A \,\dot{\alpha}\, C) \;\beta\; (B \,\dot{\alpha}\, D)}$$

$$\check{\vee} = \check{\wedge} = \vee$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\check{\boldsymbol{\alpha}} = \hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$$

$$\alpha, \beta \in \{\vee, \wedge, \mathbf{a}, \mathbf{b}, \mathbf{c}, \dots\}$$