

TOTALLY LINEAR PROOFS FOR CLASSICAL LOGICS

Alessio Guglielmi

joint work with Chris Barrett and Victoria Barrett

Proof Society 15/12/21

Talk available from AG's home page and at <https://people.bath.ac.uk/ag248/t/TLPCL.pdf>
All about deep inference at <http://alessio.guglielmi.name/res/cos>

EXAMPLE

$$(\bar{a} \wedge \bar{b}) \vee (a \vee b)$$

If $a=b=0$ the conjunction is true, otherwise the disjunction is true.

EXAMPLE

$$S_1 \equiv (\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)$$

$$\begin{aligned} S_2 \equiv & (\bar{a}_2 \wedge \bar{b}_2) \\ & \vee ((a_2 \vee b_2) \wedge \bar{a}_1) \wedge \\ & ((a_2 \vee b_2) \wedge \bar{b}_1) \\ & \vee \quad (a_1 \vee b_1) \end{aligned}$$

If $a_2 = b_2 = 0$ the first conjunction is true,
otherwise $S_2 = S_1$.

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If $a_3 = b_3 = 0$ the first conjunction is true,
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This case analysis is natural.

EXAMPLE: STATMAN

TAUTOLOGIES

Definition We call Statman tautologies the formulae

S_1, S_2, \dots :

$$S_1 \equiv (\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1) \quad \text{for } n > k \geq 1$$

$S_n \equiv$

$$(\bar{a}_n \wedge \bar{b}_n) \vee ((A_{n-1}^n \wedge B_{n-1}^n) \vee \dots \vee (A_1^n \wedge B_1^n)) \vee (a_1 \vee b_1)$$

where: $A_k^n \equiv (a_n \vee b_n) \wedge \dots \wedge (a_{k+1} \vee b_{k+1}) \wedge \bar{a}_k$
 $(\equiv (a_n \vee b_n) \wedge A_{k-1}^{n-1} \text{ if } n-1 > k)$

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We work modulo associativity.

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...

In tree-like cut-free Gentzen systems, all proofs of Statman tautologies grow at least exponentially.

[Statman, Ann. Math. Logic, 1978]

Case analysis by cut yields polynomial proofs.

EXAMPLE: STATMAN TAUTOLOGIES

Claim There exist cut-free proofs of Statman tautologies of size $O(n^{2.5})$ on the size n of the tautologies.

$$S_1 \equiv (\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1) \quad \text{for } n > k \geq 1$$

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Idea We build a cut-free derivation

$$\vdash \frac{\vdash S_1}{\vdash \vdots} \frac{\vdash \vdots}{\vdash S_n}$$

Each step must be a cut-free case analysis.

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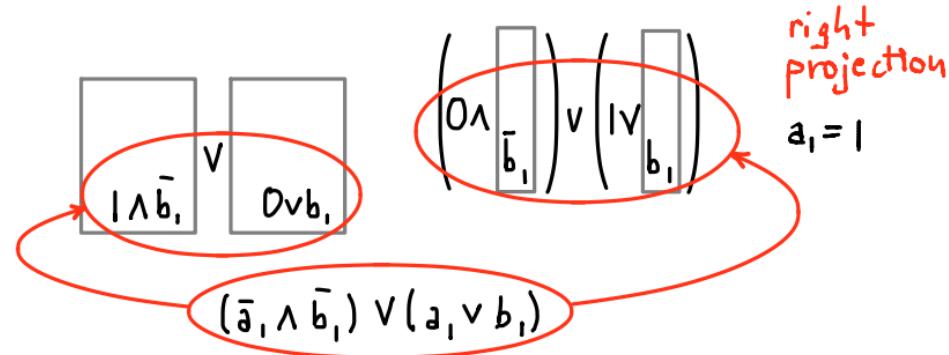
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Idea

Base case



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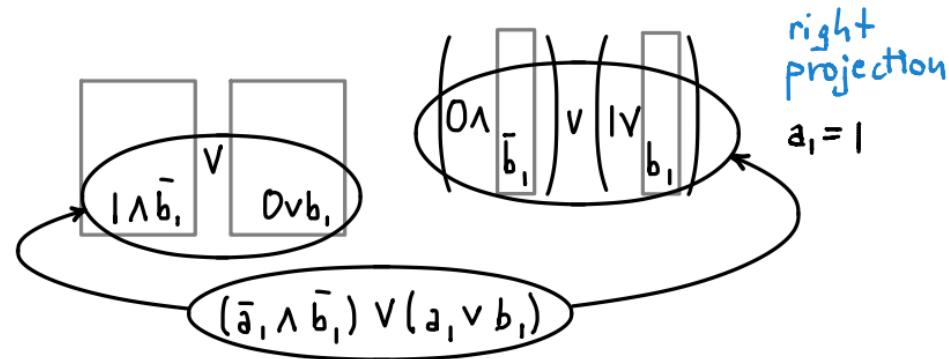
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a_i stands for $(0a_i; 1)$ (whence 'subatomic')
 b_i stands for $(0b_i; 1)$

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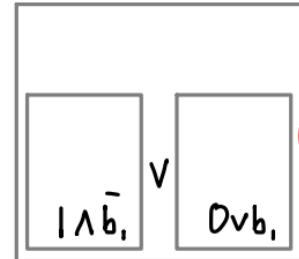
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Idea

decision tree constructor



$$(\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)$$

proofs are freely composed by connectives

$$a_1 \left(\left(0 \wedge \bar{b}_1 \right) \vee \left(1 \vee b_1 \right) \right)$$

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Idea

projection
for b_i

base case

$$\begin{aligned}
 &= (\mathbf{I} \vee \mathbf{0}) \mathbf{b}_1 (\mathbf{0} \vee \mathbf{I}) \\
 &= \frac{\bar{b}_1}{\mathbf{I} \wedge \bar{b}_1} \vee = \frac{b_1}{\mathbf{0} \vee b_1} \mathbf{d}_1 \left(\left(\mathbf{0} \wedge \frac{\mathbf{0}}{\bar{b}_1} \right) \vee \left(\mathbf{I} \vee \frac{\mathbf{0}}{b_1} \right) \right) \\
 &\quad \parallel \\
 &(\bar{a}_1 \wedge \bar{b}_1) \vee (a_1 \vee b_1)
 \end{aligned}$$

\mathbf{d}_1 stands for $(0a_1; I)$ (whence 'subatomic')
 \mathbf{b}_1 stands for $(0b_1; I)$
 \parallel are derivations to be specified (see later).
Natural case analysis.

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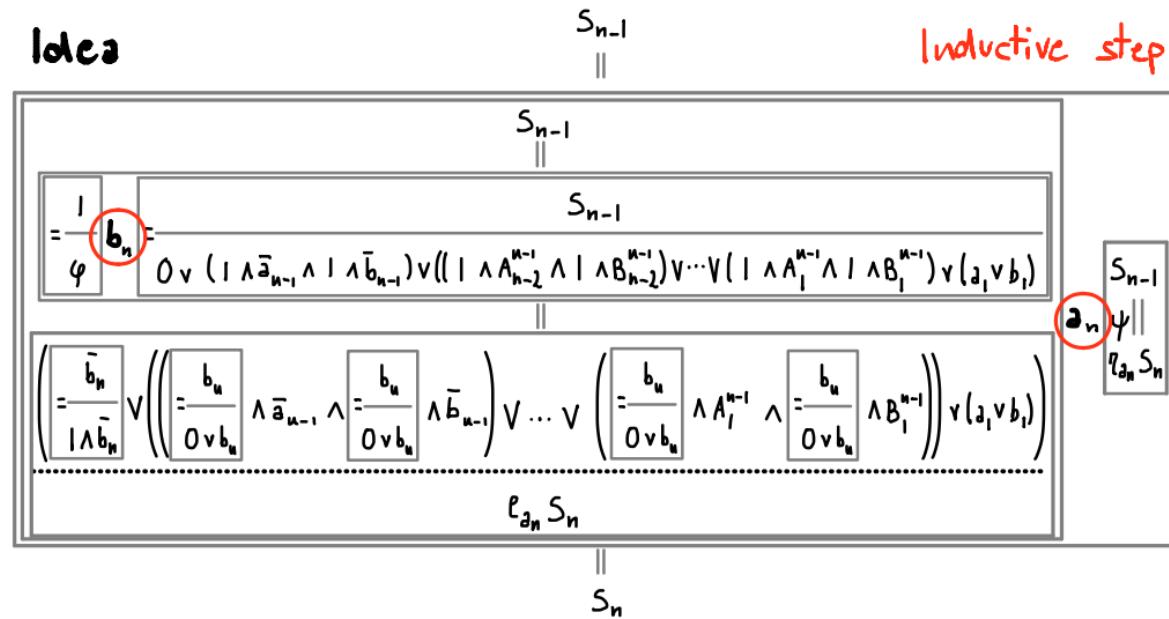
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Idea



Inductive step

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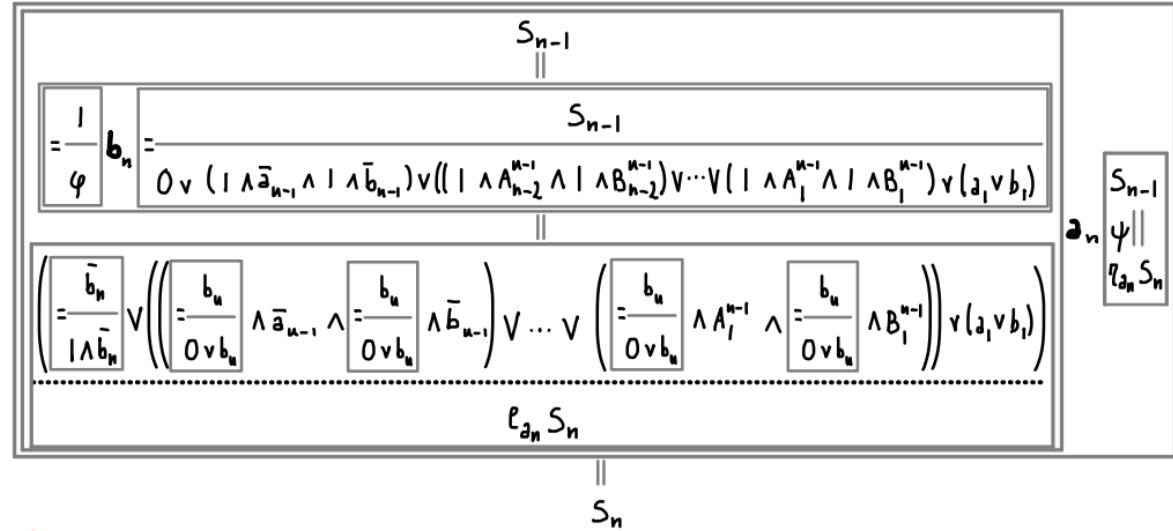
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Idea

S_{n-1}

Inductive step



where:

$$\varphi \equiv 1 \vee \boxed{(0 \wedge \bar{a}_{n-1} \wedge 0 \wedge \bar{b}_{n-1}) \vee ((0 \wedge A_{n-2}^{n-1} \wedge 0 \wedge B_{n-2}^{n-1}) \vee \dots \vee (0 \wedge A_1^{n-1} \wedge 0 \wedge B_1^{n-1})) \vee (a_1 \vee b_1)}$$

$$\psi \equiv \boxed{(0 \wedge \bar{a}_{n-1} \wedge \bar{b}_{n-1}) \vee ((A_{n-2}^{n-1} \wedge B_{n-2}^{n-1}) \vee \dots \vee (A_1^{n-1} \wedge B_1^{n-1})) \vee (a_1 \vee b_1)}$$

$$= \boxed{\left(0 \wedge \boxed{0 \parallel \bar{b}_n}\right) \vee \left(\left(1 \vee \boxed{0 \parallel b_n}\right) \wedge \bar{a}_{n-1} \wedge \left(1 \vee \boxed{0 \parallel b_n}\right) \wedge \bar{b}_{n-1}\right) \vee \dots \vee \left(\left(1 \vee \boxed{0 \parallel b_n}\right) \wedge A_1^{n-1} \wedge \left(1 \vee \boxed{0 \parallel b_n}\right) \wedge B_1^{n-1}\right) \vee (a_1 \vee b_1)}$$

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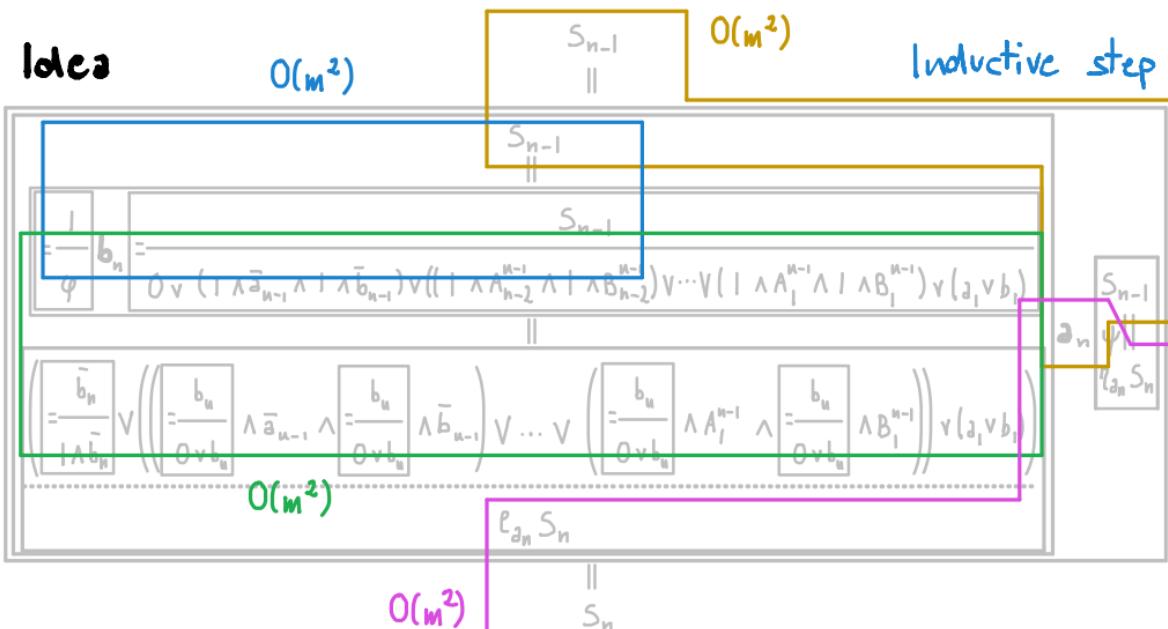
Idea

$O(m^2)$

S_{n-1}
||

$O(m^2)$

Inductive step



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$$= \boxed{\left(0 \wedge \begin{array}{|c|}\hline 0 \\ \hline \bar{b}_n \\ \hline \end{array}\right) \vee \left(\left(1 \vee \begin{array}{|c|}\hline 0 \\ \hline b_n \\ \hline \end{array}\right) \wedge \bar{a}_{n-1} \wedge \left(1 \vee \begin{array}{|c|}\hline 0 \\ \hline \bar{b}_{n-1} \\ \hline \end{array}\right) \wedge \bar{b}_{n-1}\right) \vee \dots \vee \left(\left(1 \vee \begin{array}{|c|}\hline 0 \\ \hline b_1 \\ \hline \end{array}\right) \wedge A_1^{n-1} \wedge \left(1 \vee \begin{array}{|c|}\hline 0 \\ \hline \bar{b}_1 \\ \hline \end{array}\right) \wedge B_1^{n-1}\right) \vee (a_1 \vee b_1)}$$

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Proof

Implements the idea in the proof system defined in the following.

CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

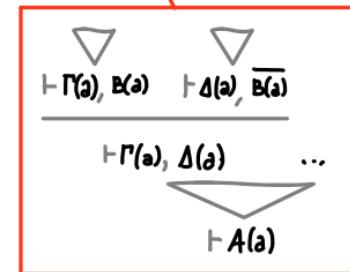
Theorem Given a proof of A , we can build a cut-tree proof of A .

CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

Theorem Given a proof of A , we can build a cut-free proof of A .

$\frac{1}{\varphi \parallel A}$ is the given derivation

e.g., obtained from
this sequent proof

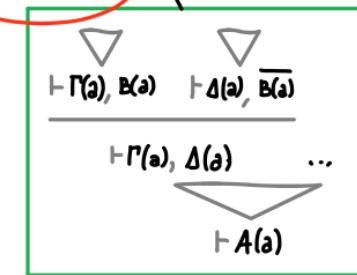


CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

Theorem Given a proof of A , we can build a cut-free proof of A .

$\frac{1}{\varphi \parallel A}$ is the given derivation and a an atom appearing in a cut instance

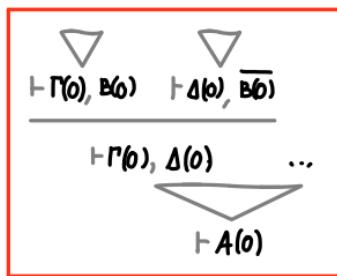
'subatomic'
 $\frac{0a1 \leftarrow a \quad 1a0 \leftarrow \bar{a}}{1a0}$
e.g., obtained from
this sequent proof



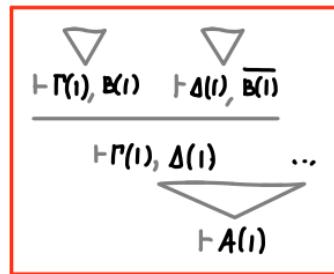
CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

Theorem Given a proof of A , we can build a cut-free proof of A .

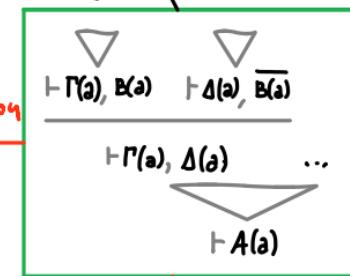
$\varphi \parallel A$ is the given derivation and a an atom appearing in a cut instance



left projection

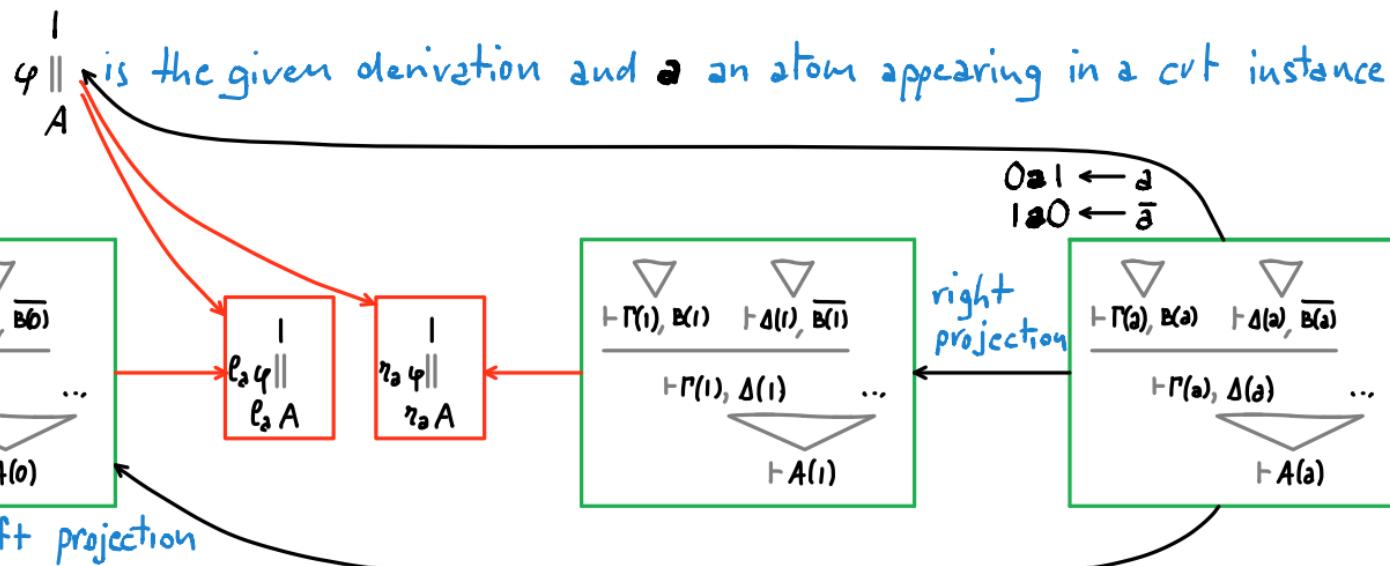


right projection



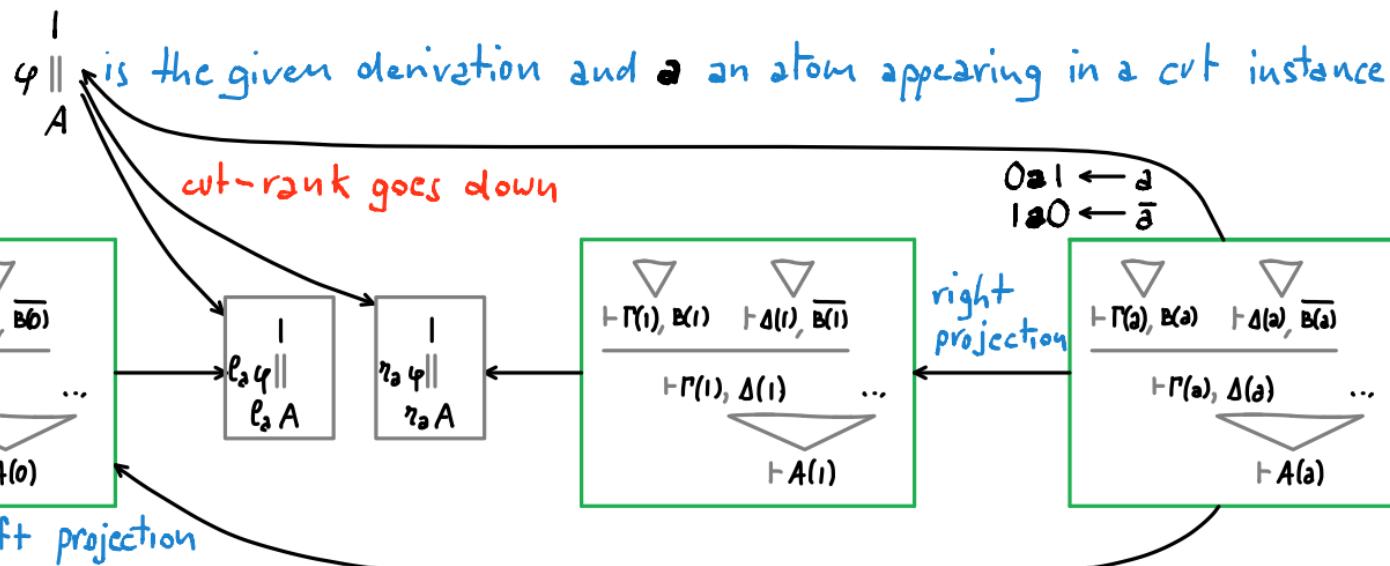
CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

Theorem Given a proof of A , we can build a cut-free proof of A .



CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

Theorem Given a proof of A , we can build a cut-free proof of A .

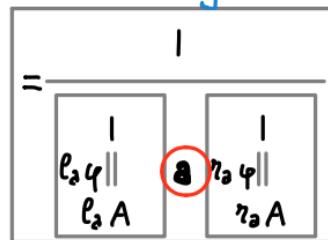


CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

Theorem Given a proof of A , we can build a cut-free proof of A .

$\frac{1}{\varphi \parallel}$ is the given derivation and a an atom appearing in a cut instance
 A

cut-rank goes down



decision trees:

$$(180) b \mid = \begin{cases} 1 & \text{if } b \\ \bar{a} & \text{if } \bar{b} \end{cases}$$

CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

Theorem Given a proof of A , we can build a cut-free proof of A .

$\varphi \parallel A$ is the given derivation and a an atom appearing in a cut instance

cut-rank goes down



no cuts

repeated applications
+ contractions

decision trees:

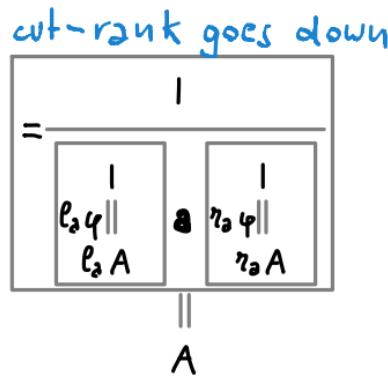
$$(1\otimes 0)b\mid = \begin{cases} 1 & \text{if } b \\ 0 & \text{if } \bar{b} \end{cases}$$

$$\frac{(A \beta B) \otimes (C \beta D)}{(A \otimes C) \wedge (B \otimes D)}$$

CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

Theorem Given a proof of A , we can build a cut-free proof of A .

$\frac{1}{\varphi \parallel}$ is the given derivation and a an atom appearing in a cut instance
 A



decision trees:

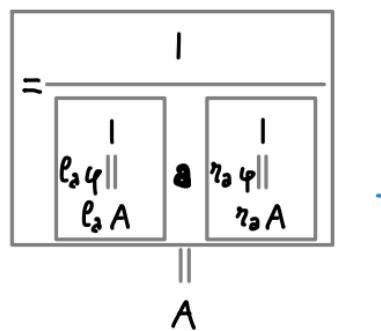
$$(180) b \mid = \begin{cases} 1 & \text{if } b \\ \bar{a} & \text{if } \bar{b} \end{cases}$$

This is free of cuts in a . Repeat.

CUT ELIMINATION VIA SUBATOMIC PROJECTIONS IN PROPOSITIONAL LOGIC WITH DECISION TREES

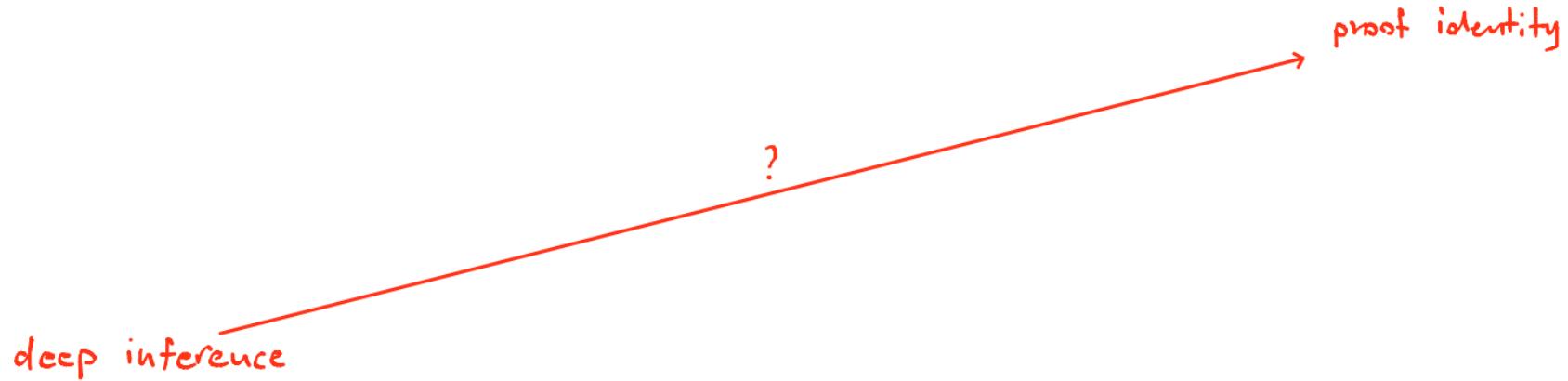
Theorem Given a proof of A , we can build a cut-free proof of A .

Proof If $\frac{1}{\varphi \parallel A}$ is the given derivation and a an atom appearing in a cut instance, build



This is free of cuts in a . Repeat.

TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS



TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS

deep inference

tool for normalisation and composition

proof identity
problem: non-trivial
equivalence classes
of proofs, taking
care of complexity

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proof semantics



past

future

deep inference



tool for normalisation and composition

proof identity

problem: non-trivial equivalence classes of proofs, taking care of complexity

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problem: non-trivial equivalence classes of proofs, taking care of complexity

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Can we check an instance of this Gentzen cut in constant time?

$$\text{cut} \frac{\vdash P, A \quad \vdash P, \bar{A}}{\vdash P}$$

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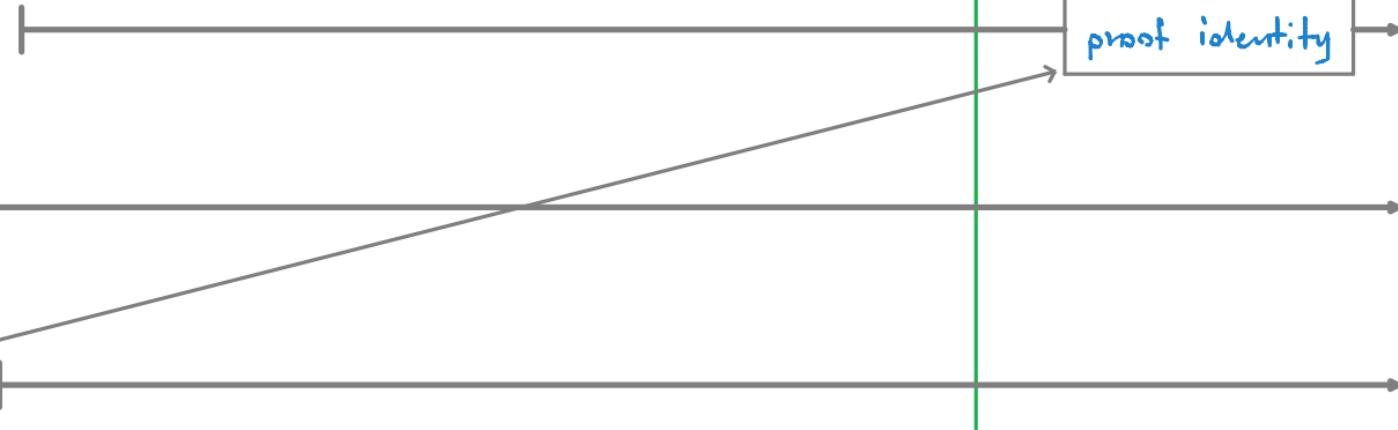


Can we check an instance of this Gentzen cut in constant time? No.
I: negation, A is unbounded

$$\text{cut} \frac{\vdash P, A \quad \vdash P, \bar{A}}{\vdash P}$$

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deep inference

$$\text{cut} \frac{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A}}{\vdash \Gamma}$$

$\vdash \Gamma$

Can we check an instance of this Gentzen cut in constant time? No.

I: negation, A is unbounded;

II: contraction, Γ is unbounded.

We say that the cut is non-local.

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complexity



deep inference



non-local

$$\boxed{\begin{array}{c} \text{Gentzen} \\ \frac{\Gamma, A \quad \Gamma, \bar{A}}{\Gamma} \end{array}}$$

Can we check an instance of this Gentzen cut in constant time? No.

1: negation, A is unbounded;

2: contraction, Γ is unbounded.

We say that the cut is non-local.

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deep inference



non-local

$$\boxed{\text{Gentzen}} \quad \frac{\text{cut}}{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A} \quad \vdash \Gamma}$$

$$\text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

linear

$$\frac{\text{cut}}{a \wedge \bar{a}}$$

atomic

$$\frac{\text{cut}}{0}$$

linear or atomic = local

negation, A is unbounded
contraction, Γ is unbounded

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non-local

$$\boxed{\text{Gentzen}} \quad \frac{\Gamma \vdash A \quad \Gamma \vdash \bar{A}}{\Gamma \vdash}$$

(The Gentzen rule for non-locality, showing that if both A and \bar{A} are derivable, then the empty set is derivable.)

$$\text{switch} \quad \frac{\overbrace{(A_1 \vee A_2)}^A \wedge (\bar{A}_1, \bar{A}_2)}{(A_1, \bar{A}_1) \vee (A_2, \bar{A}_2)}$$

(A switch rule for disjunction, where the formula A is split into $A_1 \vee A_2$ and the conjunction of their negations.)

*cut formulae
are reduced*

locality

atomic

$$\text{cut} \quad \frac{a \wedge \bar{a}}{0}$$

(A cut rule for atomic formulas, showing that if a and \bar{a} are both present, the derivation reduces to 0.)

linear or atomic = local

*negation, A is unbounded
contraction, Γ is unbounded*

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non-local

$$\boxed{\text{Gentzen}} \quad \frac{\text{cut}}{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A} \quad \vdash \Gamma}$$

switch $\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$ medial $\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$	linear $\frac{}{a \wedge \bar{a}}$ cut $\frac{}{0}$	atomic $\frac{\text{dt. contr.}}{a}$
--	--	--

linear or atomic = local

negation, A is unbounded
contraction, Γ is unbounded

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non-local

$$\text{Gentzen} \quad \frac{\text{cut}}{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A}} \quad \vdash \Gamma$$

contractions are reduced

$$\begin{array}{c} \text{linear} \\ \text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)} \\ \text{medial} \frac{(\Gamma, \wedge, \Gamma_2) \vee (\Gamma, \wedge, \Gamma_2)}{(\Gamma, \vee \Gamma_1) \wedge (\Gamma_2, \vee \Gamma_2)} \end{array}$$

atomic

$$\frac{\text{dt. cut}}{a \wedge \bar{a}} \quad \frac{\text{0}}{0}$$

$$\frac{\text{dt. contr.}}{a \vee a} \quad \frac{a}{a}$$

linear or atomic = local

negation, A is unbounded

contraction, Γ is unbounded

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non-local

Cutzen

$$\frac{\text{cut}}{\vdash \Gamma} \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A}}{A}$$

$$\text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\frac{\text{dt. cut}}{\text{cut}} \frac{a \wedge \bar{a}}{0}$$

deep
inference

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt. contr.}}{\text{contr.}} \frac{a \vee \bar{a}}{a}$$

local

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[Bruscoli, Guglielmi, ACM ToCL 2009]

complexity

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$$\text{Gentzen} \quad \frac{\text{cut}}{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A}} \vdash \Gamma$$

local

$$\text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\frac{\text{dt. cut}}{a \wedge \bar{a}} \quad 0$$

deep inference

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt. contr.}}{a \vee \bar{a}} \quad a$$

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[Jeřábek, JLC 2009]

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$$\frac{\text{cut}}{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A}} \vdash \Gamma$$

$$\text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\frac{\text{dt. cut}}{a \wedge \bar{a}} \quad 0$$

deep inference

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt. contr.}}{a \vee \bar{a}} \quad a$$

local

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[Gundersen Heijltjes,
Parigot, LICS 2013]

atomic λ -calculus -
fully lazy sharing

proof identity

local

non-local

$$\frac{\text{cut}}{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A}} \vdash \Gamma$$

$$\text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\frac{\text{dt. cut}}{a \wedge \bar{a}} \quad 0$$

deep
inference

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt. contr.}}{a \vee \bar{a}} \quad a$$

TIMELINE, LOCALITY AND SUBATOMIC SYSTEMS

[Hughes, Straßburger, Wu, LICS 2021]

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$$\frac{\text{Gentzen} \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, \bar{A}}{\text{cut}}}{\vdash \Gamma}$$

$$\text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\frac{\text{dt.} \quad a \wedge \bar{a}}{\text{cut} \quad 0}$$

deep
inference

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt.} \quad a \vee a}{\text{contr.} \quad a}$$

local

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Gentzen

$$\frac{\text{cut}}{\vdash \Gamma}$$

switch

$$\frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

medial

$$\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

at. cut

$$\frac{a \wedge \bar{a}}{0}$$

at. contr.

$$\frac{a \vee \bar{a}}{a}$$

deep
inference

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$$\text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\frac{\text{dt.} \quad a \wedge \bar{a}}{\text{cut} \quad 0}$$

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt.} \quad a \vee \bar{a}}{\text{contr.} \quad \bar{a}}$$

$$\frac{\text{shape} \quad (A \beta B) \alpha (C \hat{\beta} D)}{\text{saturation} \quad (A \alpha C) \beta (B \alpha D)}$$

$$\hat{\vee} = \hat{\wedge} = \wedge$$

$$\alpha, \beta \in \{\vee, \wedge\}$$

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$$\text{switch} \frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\frac{\text{dt.} \frac{a \wedge \bar{a}}{0}}{\text{cut}}$$

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt.} \frac{a \vee \bar{a}}{a}}{\text{contr.} \frac{}{a}}$$

all instances
are sound

$$\begin{array}{ccc} \text{shape} & & \text{saturation} \\ \frac{(A \beta B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)} & & \check{V} = \check{\lambda} = V \\ & & \hat{V} = \hat{\lambda} = \Lambda \\ \alpha, \beta \in \{ \vee, \wedge \} & & \end{array}$$

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$$\text{switch} \frac{(A \vee B) \wedge (C \vee D)}{(A \vee C) \vee (B \wedge D)}$$

$$\frac{\text{dt.} \frac{a \wedge \bar{a}}{0}}{\text{cut}}$$

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt.} \frac{a \vee a}{a}}{\text{contr.} \frac{}{a}}$$

all instances
are sound

$$\begin{array}{ccc} \text{shape} & & \text{satisfaction} \\ \frac{(A \beta B) \alpha (C \beta D)}{(A \checkmark C) \beta (B \alpha D)} & & \check{v} = \check{\lambda} = v \\ & & \hat{v} = \hat{\lambda} = \wedge \end{array}$$

$$\alpha, \beta \in \{ \vee, \wedge \}$$

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$$\text{switch} \frac{(A \wedge B) \wedge (C \vee D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\frac{\text{dt.} \frac{a \wedge \bar{a}}{0}}{\text{cut}}$$

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt.} \frac{a \vee \bar{a}}{a}}{\text{contr.} \frac{}{a}}$$

all instances
are sound

$$\begin{array}{ccc} \text{shape} & & \text{satisfaction} \\ \frac{(A \hat{\beta} B) \alpha (C \beta D)}{(A \alpha C) \beta (B \alpha D)} & & \check{v} = \check{\lambda} = v \\ & & \hat{v} = \hat{\lambda} = \wedge \end{array}$$

$$\alpha, \beta \in \{\vee, \wedge\}$$

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$$\text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\frac{\text{dt.} \frac{a \wedge \bar{a}}{0}}{\text{cut}}$$

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt.} \frac{a \vee \bar{a}}{a}}{\text{contr.} \frac{}{a}}$$

shape

$$(A \dot{\beta} B) \star (C \dot{\beta} D)$$

saturation

$$\check{v} = \check{\lambda} = v$$

$$(A \dot{\alpha} C) \beta (B \dot{\alpha} D)$$

$$\hat{v} = \hat{\lambda} = \lambda$$

one corner is saturated

$$\alpha, \beta \in \{ \vee, \wedge \}$$

}

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$$\text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\frac{\text{dt.}}{\text{cut}} \frac{a \wedge \bar{a}}{0}$$

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt. contr.}}{} \frac{a \vee \bar{a}}{a}$$

What about
atomic rules?

$$\frac{\text{shape}}{(A \dot{\beta} B) \star (C \dot{\beta} D)} \quad \frac{\text{saturation}}{\hat{V} = \hat{\lambda} = V}$$

$$\frac{(A \dot{\alpha} C) \dot{\beta} (B \dot{\alpha} D)}{\hat{V} = \hat{\lambda} = \Lambda}$$

$$\alpha, \beta \in \{V, \wedge\}$$

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$$\text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\frac{\text{dt.} \quad a \wedge \bar{a}}{\text{cut} \quad 0}$$

instance of the shape

$$\frac{(A \otimes B) \wedge (C \oslash D)}{(A \wedge C) \otimes (B \oslash D)}$$

shape

$$\frac{(A \dot{\wedge} B) \times (C \dot{\wedge} D)}{(A \dot{\wedge} C) \dot{\wedge} (B \dot{\wedge} D)}$$

saturation

$$\check{V} = \check{\lambda} = V$$

$$\hat{V} = \hat{\lambda} = \Lambda$$

$$\check{\alpha} = \hat{\alpha} = \alpha$$

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt.} \quad a \vee a}{\text{contr.} \quad \bar{a}}$$

self-dual non-commutative 'atoms'
 $\alpha, \beta \in \{V, \wedge, \dot{\wedge}, \otimes, \oslash, \dot{\wedge}, \times, \dot{\wedge}, \vee, \wedge, \dot{\wedge}, \otimes, \oslash, \dot{\wedge}, \times, \dot{\wedge}, \dots\}$

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$$\text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\frac{\text{dt. cut}}{0}$$

instance of the shape

$$\frac{(0 \alpha 1) \wedge (1 \beta 0)}{(0 \wedge 1) \alpha (1 \beta 0)}$$

$$\frac{(0 \wedge 1) \alpha (1 \beta 0)}{(0 \alpha 1) \wedge (1 \beta 0)}$$

shape

$$\frac{(A \dot{\beta} B) \times (C \dot{\beta} D)}{(A \dot{\alpha} C) \beta (B \dot{\alpha} D)}$$

saturation

$$\check{V} = \check{\lambda} = V$$

$$\hat{V} = \hat{\lambda} = \wedge$$

$$\check{\alpha} = \hat{\alpha} = \alpha$$

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt. contr.}}{2}$$

superposition of truth
values

$$\alpha, \beta \in \{V, \wedge, \alpha, \beta, c, \dots\}$$

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$$\text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

$$\frac{\text{dt. cut}}{0}$$

instance of the shape

$$\frac{(0 \alpha 1) \wedge (1 \beta 0)}{(0 \wedge 1) \alpha (1 \wedge 0)}$$

shape

$$\frac{(A \dot{\wedge} B) \times (C \dot{\wedge} D)}{(A \dot{\wedge} C) \dot{\wedge} (B \dot{\wedge} D)}$$

saturation

$$\check{V} = \check{\lambda} = V$$

$$\hat{V} = \hat{\lambda} = \Lambda$$

$$\check{\alpha} = \hat{\alpha} = \alpha$$

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt. contr.}}{a}$$

interpretation - CL

$$\alpha, \beta \in \{V, \wedge, \alpha, b, c, \dots\}$$

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instances of the shape

$$\text{switch} \frac{(A \vee B) \wedge (C \wedge D)}{(A \wedge C) \vee (B \wedge D)}$$

instance of the shape

$$\frac{(0 \otimes 1) \wedge (1 \otimes 0)}{(0 \wedge 1) \oplus (1 \wedge 0)}$$

shape

$$\frac{(A \dot{\wedge} B) \times (C \dot{\wedge} D)}{(A \dot{\wedge} C) \dot{\wedge} (B \dot{\wedge} D)}$$

saturation

$$\begin{aligned} \check{V} &= \check{\lambda} = V \\ \hat{V} &= \hat{\lambda} = \Lambda \\ \check{\alpha} &= \hat{\alpha} = \alpha \end{aligned}$$

$$\text{medial} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

$$\frac{\text{dt. contr.} \frac{a \vee a}{a}}{(0 \otimes 1) \vee (0 \otimes 1)} \longleftrightarrow \frac{(0 \otimes 1) \vee (0 \otimes 1)}{(0 \vee 0) \oplus (1 \vee 1)}$$

interpretation - CL

$$\alpha, \beta \in \{V, \wedge, \oplus, \otimes, \dots\}$$

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instances of the shape

$$\text{switch} \frac{(A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

$$\text{medial} \frac{(A \otimes B) \otimes (C \otimes D)}{(A \otimes C) \otimes (B \otimes D)}$$

instance of the shape

$$\frac{(\perp \otimes \top) \otimes (\top \otimes \perp)}{(\perp \otimes \top) \oplus (\top \otimes \perp)}$$

$$\frac{\text{dt.}}{\text{cut}} \frac{a \otimes \bar{a}}{0}$$

$$\frac{\text{dt. contr.}}{a} \frac{a \otimes a}{a}$$

shape

$$\frac{(A \ddot{\otimes} B) \times (C \ddot{\otimes} D)}{(A \ddot{\otimes} C) \ddot{\otimes} (B \ddot{\otimes} D)}$$

interpretation - LL

$$\alpha, \beta \in \{\otimes, \otimes, a, b, c, \dots\}$$

$\otimes = \ddot{\otimes} = \otimes$

$\hat{\otimes} = \ddot{\otimes} = \otimes$

$\ddot{\alpha} = \hat{\alpha} = \alpha$

$\hat{\alpha} = \alpha$

$\alpha = \alpha$

$\beta = \beta$

$\gamma = \gamma$

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1 study subatomic
normalisation

2 interpret it
for different logics
(CL, LL, BV, ...)

[Aler Tubella, Guglielmi,
ACM ToCL, 2018]

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standard deep inference

$$(A \vee B) \wedge (C \wedge D)$$

$$a \wedge \bar{a}$$

$$(A \wedge C) \vee (B \wedge D)$$

$$0$$

$$(A \wedge B) \vee (C \wedge D)$$

$$a \vee a$$

$$(A \vee C) \wedge (B \vee D)$$

$$a$$

subatomic
proof
systems

$$(A \dot{\beta} B) \star (C \dot{\beta} D)$$

shape

$$\check{v} = \check{\lambda} = v$$

$$(A \dot{\alpha} C) \beta (B \dot{\alpha} D)$$

$\hat{v} = \hat{\lambda} = \wedge$

+ unit
equations

$$\check{\alpha} = \hat{\alpha} = \alpha$$

$$\alpha, \beta \in \{V, \wedge, a, b, c, \dots\}$$

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the shape works for unary connectives

$$(\dot{\beta} A) \alpha (\dot{\beta} B)$$

$$\beta(A \dot{\alpha} B)$$

$$\alpha(A \dot{\beta} B)$$

$$(\dot{\alpha} A) \beta (\dot{\alpha} B)$$

$$\text{shape} \quad (A \dot{\beta} B) \alpha (C \dot{\beta} D)$$

$$(A \dot{\alpha} C) \beta (B \dot{\alpha} D)$$

+ unit
equations

$$\alpha, \beta \in \{V, \wedge, \alpha, \vee, \neg, \dots\}$$

saturation

$$\check{V} = \check{\lambda} = V$$

$$\hat{V} = \hat{\lambda} = \Lambda$$

$$\check{\alpha} = \hat{\alpha} = \alpha$$

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the shape works for unary connectives

$$\diamond A \wedge \square B$$

$$\diamond(A \wedge B)$$

$$\square(A \vee B)$$

$$\square A \vee \diamond B$$

$$(\dot{\beta} A) \alpha (\dot{\beta} B)$$

$$\beta(A \dot{\alpha} B)$$

$$\alpha(A \dot{\beta} B)$$

$$(\dot{\alpha} A) \beta (\dot{\alpha} B)$$

$$(A \dot{\beta} B) \alpha (C \dot{\beta} D)$$

$$(A \dot{\alpha} C) \beta (B \dot{\alpha} D)$$

+ unit
equations

$$\alpha, \beta \in \{V, \wedge, \alpha, \vee, \dots\}$$

$$\check{V} = \check{\lambda} = V$$

$$\hat{V} = \hat{\lambda} = \wedge$$

$$\check{\alpha} = \hat{\alpha} = \alpha$$

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the shape works for unary connectives

$$\exists x. A \wedge \forall x. B$$

$$\exists x. (A \wedge B)$$

$$\forall x. (A \vee B)$$

$$\forall x. A \vee \exists x. B$$

$$(\dot{\beta} A) \alpha (\dot{\beta} B)$$

$$\beta (A \dot{\alpha} B)$$

$$\alpha (A \dot{\beta} B)$$

$$(\dot{\alpha} A) \beta (\dot{\alpha} B)$$

shape	saturation
$(A \dot{\beta} B) \alpha (C \dot{\beta} D)$	$\check{V} = \check{\lambda} = V$
$(A \dot{\alpha} C) \beta (B \dot{\alpha} D)$	$\hat{V} = \hat{\lambda} = \Lambda$
+ unit equations	$\check{\alpha} = \hat{\alpha} = \alpha$

$\alpha, \beta \in \{V, \wedge, \exists, \vee, \forall, \dots\}$

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$$\frac{}{\exists x. \forall y. A} \quad \text{---}$$

$$\frac{}{\alpha(\dot{\beta}A)} \quad \text{---}$$

$$\frac{}{\exists x. A \wedge \forall x. B}$$

$$\frac{}{\exists x. (A \wedge B)}$$

$$\frac{}{(\dot{\beta}A) \alpha(\dot{\beta}B)}$$

$$\frac{}{\beta(A \dot{\alpha} B)}$$

$$\frac{}{\forall x. (A \vee B)}$$

$$\frac{}{\forall x. A \vee \exists x. B}$$

$$\frac{}{\alpha(A \dot{\beta} B)}$$

$$\frac{}{(\dot{\alpha}A) \beta(\dot{\alpha}B)}$$

shape	saturation
$(A \dot{\beta} B) \alpha(C \dot{\beta} D)$	$\dot{V} = \dot{\lambda} = V$
$(A \dot{\alpha} C) \beta(B \dot{\alpha} D)$	$\dot{V} = \dot{\lambda} = \wedge$
+ unit equations	$\dot{\alpha} = \dot{\alpha} = \alpha$
	$\alpha, \beta \in \{V, \wedge, \alpha, \beta, c, \dots\}$

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the shape works for unary connectives

$$\exists x. A \wedge \forall x. B$$

$$\alpha(\dot{\beta}A)$$

$$\exists x. (A \wedge B)$$

$$(\dot{\beta}A) \alpha (\dot{\beta}B)$$

$$\forall y. \exists x. A$$

$$\beta(\dot{\alpha}A)$$

$$\forall x. (A \vee B)$$

$$\alpha(A \dot{\beta} B)$$

quantifier shifts!

$$\forall x. A \vee \exists x. B$$

$$(\dot{\alpha}A) \beta (\dot{\alpha}B)$$

shape	saturation
$(A \dot{\beta} B) \alpha (C \dot{\beta} D)$	$\dot{V} = \dot{\lambda} = V$
$(A \dot{\alpha} C) \beta (B \dot{\alpha} D)$	$\hat{V} = \hat{\lambda} = \Lambda$
+ unit equations	$\dot{\alpha} = \hat{\alpha} = \alpha$

$\alpha, \beta \in \{V, \wedge, \alpha, \vee, \beta, \dots\}$

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I Take all instances of the shape. You get:

- a proof system for propositional classical logic
with decision trees

A if a false and b false

B if a true and b false

...

invertible rule

$$(A \circledast B) \circledast (C \circledast D)$$

$$(A \bullet C) \circledast (B \bullet D)$$

shape	saturation
$(A \overset{\alpha}{\circledast} B) \circledast (C \overset{\beta}{\circledast} D)$	$\check{v} = \check{\lambda} = v$
$(A \overset{\alpha}{\bullet} C) \circledast (B \overset{\beta}{\bullet} D)$	$\hat{v} = \hat{\lambda} = \wedge$
+ unit equations	$\check{\alpha} = \hat{\alpha} = \alpha$

$\alpha, \beta \in \{v, \wedge, \circledast, \bullet, c, \dots\}$

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I Take all instances of the shape. You get:

- a proof system for propositional classical logic with decision trees;
- a simple cut-elimination procedure based on projections

shape	saturation
$(A \dot{\beta} B) \times (C \dot{\beta} D)$	$\check{v} = \check{\lambda} = v$
$(A \dot{\alpha} C) \beta (B \dot{\alpha} D)$	$\hat{v} = \hat{\lambda} = \lambda$
+ unit equations	$\check{\alpha} = \hat{\alpha} = \alpha$
$\alpha, \beta \in \{v, \lambda, a, b, c, \dots\}$	

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I Take all instances of the shape. You get:

- a proof system for propositional classical logic with decision trees;
- a simple cut-elimination procedure based on projections;
- projections build polynomial-size proofs of Statman tautologies.

[C. Barrett, Guglielmi, ACM ToCL, to appear]

shape	saturation
$(A \dot{\beta} B) \times (C \dot{\beta} D)$	$\check{V} = \check{\Lambda} = V$
$(A \dot{\alpha} C) \beta (B \dot{\alpha} D)$	$\hat{V} = \hat{\Lambda} = \Lambda$
+ unit equations	$\check{\alpha} = \hat{\alpha} = \alpha$
	$\alpha, \beta \in \{V, \Lambda, \check{\alpha}, \hat{\alpha}, \dots\}$

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2 Take the unit equations

$$\begin{array}{cccccc}
 & \frac{A}{AVO} & \frac{A}{A\Lambda I} & \frac{A}{A\wedge I} & & \\
 & AVO & A & A\Lambda I & A & \\
 \hline
 & \frac{0}{0\wedge 0} & \frac{0\otimes 0}{0} & \frac{|V|}{1} & \frac{|a|}{1} & \frac{|I|}{1\otimes 1}
 \end{array}$$

shape	saturation
$(A \dot{\beta} B) \star (C \dot{\beta} D)$	$\check{v} = \check{\lambda} = v$
$(A \dot{\alpha} C) \beta (B \dot{\alpha} D)$	$\hat{v} = \hat{\lambda} = \lambda$
+ unit equations	$\check{\alpha} = \hat{\alpha} = \alpha$
$\alpha, \beta \in \{V, \wedge, \otimes, \star, \cdot, \dots\}$	

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locality

2 Take the unit equations:

- they are admissible

$$\begin{array}{cccc}
 \frac{}{A} & \frac{}{AVO} & \frac{}{A} & \frac{}{A \wedge I} \\
 AVO & A & A \wedge I & A \\
 \frac{0}{0 \wedge 0} & \frac{0 \otimes 0}{0} & \frac{1 \vee 1}{1} & \frac{1 \otimes 1}{1} \\
 0 \wedge 0 & 0 \otimes 0 & 1 & 1 \otimes 1
 \end{array}$$

shape	saturation
$(A \dot{\beta} B) \star (C \dot{\beta} D)$	$\check{V} = \check{\lambda} = V$
$(A \dot{\alpha} C) \beta (B \dot{\alpha} D)$	$\hat{V} = \hat{\lambda} = \Lambda$
+ unit equations	$\check{\alpha} = \hat{\alpha} = \alpha$
	$\alpha, \beta \in \{V, \wedge, \otimes, \star, \vee, \wedge, \otimes, \beta, \alpha, \dots\}$

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2 Take the unit equations:

- they are admissible,
- for several logics.

$$\begin{array}{c}
 \text{shape} \\
 \frac{(A \dot{\beta} B) \times (C \dot{\beta} D)}{(A \dot{\alpha} C) \beta (B \dot{\alpha} D)} \\
 \hline
 \text{saturation} \\
 \check{v} = \check{\lambda} - v \\
 \hat{v} = \hat{\lambda} - \lambda \\
 \check{\alpha} = \hat{\alpha} - \alpha
 \end{array}$$

$$\alpha, \beta \in \{v, \wedge, \alpha, \beta, c, \dots\}$$

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locality

2 Take the unit equations:

- they are admissible,
 - for several logics.
- total linearity

[V. Barrett, Guglielmi, TLLA 2021, in preparation]

$$\frac{\text{shape} \quad (A \overset{\circ}{\beta} B) \times (C \overset{\circ}{\beta} D)}{\text{saturation} \quad (A \overset{\circ}{\alpha} C) \beta (B \overset{\circ}{\alpha} D)}$$

logic dependent

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factorisation

Take the unit equations:

- they are admissible,
 - for several logics.
- total linearity

→ proof substitution and factorisation

$$\frac{\text{shape} \quad (A \dot{\beta} B) \times (C \dot{\beta} D)}{\text{saturation} \quad (A \dot{\alpha} C) \beta (B \dot{\alpha} D)}$$

logic dependent

ADMISSIBILITY OF UNIT EQUATIONS

$$\frac{(A \dot{\beta} B) \times (C \dot{\beta} D)}{(A \dot{\alpha} C) \beta (B \dot{\alpha} D)}$$
$$\check{v} = \check{\lambda} = v$$
$$\hat{v} = \hat{\lambda} = \wedge$$
$$\check{\alpha} = \hat{\alpha} = \alpha$$

$\alpha, \beta \in \{v, \wedge, \check{a}, \check{b}, \check{c}, \dots\}$

$$\begin{array}{c} A \\ \hline AVO \end{array} \quad \begin{array}{c} AVO \\ \hline A \end{array} \quad \begin{array}{c} A \\ \hline A \wedge I \end{array} \quad \begin{array}{c} A \wedge I \\ \hline A \end{array}$$

$$\begin{array}{c} 0 \\ \hline 0 \wedge 0 \end{array} \quad \begin{array}{c} 0 \\ \hline 0 \vee 0 \end{array} \quad \begin{array}{c} 0 \vee 0 \\ \hline 0 \end{array}$$

$$\begin{array}{c} 1 \vee 1 \\ \hline 1 \end{array} \quad \begin{array}{c} 1 \vee 1 \\ \hline 1 \vee 1 \end{array} \quad \begin{array}{c} 1 \\ \hline 1 \vee 1 \end{array}$$

propositional classical logic
(with decision trees)

ADMISSIBILITY OF UNIT EQUATIONS

Lemma (1) arc admissible.

Proof Move steps up and down. Steps vanish at top and bottom. Interpretation does not change. Example:

$$\begin{array}{c}
 \begin{array}{c} 0 \otimes 0 \\ \hline 0 \end{array} \rightarrow \begin{array}{c} 0 \quad 0 \\ \hline 0 \otimes 0 \quad 0 \otimes 0 \\ \hline 0 \end{array} \\
 \begin{array}{c} 0 \\ \hline 0 \otimes 0 \end{array} \quad \begin{array}{c} 0 \otimes 0 \quad 0 \otimes 0 \\ \hline 1 \end{array} \\
 \end{array}$$

propositional classical logic
(with decision trees)

$$\begin{array}{c}
 (A \dot{\beta} B) \times (C \dot{\beta} D) \quad \check{v} = \check{\lambda} = v \\
 \hline
 (A \dot{\alpha} C) \wedge (B \dot{\alpha} D) \quad \hat{v} = \hat{\lambda} = \lambda \\
 \check{\alpha} = \hat{\alpha} = \alpha
 \end{array}$$

$$\alpha, \beta \in \{v, \lambda, a, b, c, \dots\}$$

$$\begin{array}{c}
 \begin{array}{c} A \quad A \vee 0 \quad A \quad A \wedge 1 \\ \hline A \vee 0 \quad A \quad A \wedge 1 \quad A \end{array}
 \end{array}$$

(1)

$$\begin{array}{c}
 \boxed{\begin{array}{c} 0 \quad 0 \quad 0 \otimes 0 \\ \hline 0 \otimes 0 \quad 0 \otimes 0 \quad 0 \\ \hline 1 \vee 1 \quad 1 \otimes 1 \quad 1 \\ \hline 1 \quad 1 \quad 1 \otimes 1 \end{array}}
 \end{array}$$

ADMISSIBILITY OF UNIT EQUATIONS

Lemma (1) are admissible.

Lemma (Eversion) Given A and B , there exist

$$\begin{array}{c} [B^i \Rightarrow x_i]_{\underline{\lambda}} \check{A} \\ \parallel \\ [\check{A}^j \Rightarrow y_j]_{\underline{\delta}} B \end{array}$$

where B^i 's (resp. \check{A}^j 's) are renamings of B (resp. \check{A}) and $\forall_i B^i = \exists_j \check{A}^j$.

$$\frac{(A \dot{\beta} B) \times (C \dot{\beta} D)}{(A \dot{\alpha} C) \wedge (B \dot{\alpha} D)}$$

$\check{v} = \check{\lambda} = v$
 $\hat{v} = \hat{\lambda} = \Lambda$
 $\check{\alpha} = \hat{\alpha} = \alpha$

$$\alpha, \beta \in \{v, \Lambda, \check{a}, b, c, \dots\}$$

$$\frac{A}{AVO} \quad \frac{AVO}{A} \quad \frac{A}{A \wedge I} \quad \frac{A \wedge I}{A}$$

(1)

0	0	0 a 0
0 a 0	0 a 0	0
1 v 1	1 a 1	1
1	1	1 a 1

ADMISSIBILITY OF UNIT EQUATIONS

Lemma (i) are admissible.

Lemma (Eversion) Given A and B , there exist

substitute into variables
of B $\frac{[B^i \Rightarrow x_i] \underline{\lambda} \check{A}}{[A^j \Rightarrow y_j] \underline{\delta} B}$. A connectives saturated down

where B^i 's (resp. \check{A}^j 's) are renamings of B (resp. \check{A}) and $U_i B^i = U_j \check{A}^j$.

Proof Use the shape in a straightforward induction.

$$\frac{(A \dot{\beta} B) \times (C \dot{\beta} D)}{(A \dot{\alpha} C) \dot{\beta} (B \dot{\alpha} D)}$$

$$\check{v} = \check{\lambda} = v$$

$$\hat{v} = \hat{\lambda} = \Lambda$$

$$\check{\alpha} = \hat{\alpha} = \alpha$$

$$\alpha, \beta \in \{v, \Lambda, a, b, c, \dots\}$$

$$\frac{A}{AVO} \quad \frac{AVO}{A} \quad \frac{A}{A\Lambda I} \quad \frac{A\Lambda I}{A}$$

(i)

$$\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0\Lambda 0 & 0\alpha 0 & 0 \\ \hline 1\vee 1 & 1\alpha 1 & 1 \\ \hline 1 & 1 & 1\alpha 1 \end{array}$$

ADMISSIBILITY OF UNIT EQUATIONS

Lemma (1) are admissible.

Lemma (Eversion) Given A and B , there exist

$$\begin{array}{c} [B^i \Rightarrow x_i]_{\underline{\lambda}} \check{A} \\ \parallel \\ [\check{A}^j \Rightarrow y_j]_{\underline{\beta}} B \end{array}$$

where B^i 's (resp. \check{A}^j 's) are renamings of B (resp. \check{A}) and $U_i B^i = U_j \check{A}^j$.

Theorem (2) are admissible.

$$\begin{array}{c} (A \dot{\beta} B) \times (C \dot{\beta} D) \\ \hline (A \dot{\alpha} C) \wedge (B \dot{\alpha} D) \end{array} \quad \begin{array}{l} \check{v} = \check{\lambda} = v \\ \hat{v} = \hat{\lambda} = \Lambda \\ \check{\alpha} = \hat{\alpha} = \alpha \end{array}$$

(2) $\alpha, \beta \in \{v, \Lambda, \check{a}, b, c, \dots\}$

A	$A \vee 0$	A	$A \wedge I$
$A \vee 0$	A	$A \wedge I$	A

(1)	0	0	0 ≥ 0
	0 $\wedge 0$	0 ≥ 0	0
	1 $\vee 1$	1 ≥ 1	1
	1	1	1 ≥ 1

ADMISSIBILITY OF UNIT EQUATIONS

Lemma (1) are admissible.

Lemma (Eversion) Given A and B , there exist

$$\begin{array}{c} [B^i \Rightarrow x_i]_A \check{A} \\ \parallel \\ [\check{A}^j \Rightarrow y_j]_B B \end{array}$$

where B^i 's (resp. \check{A}^j 's) are renamings of B (resp. \check{A}) and $U_i B^i = U_j \check{A}^j$.

Theorem (2) are admissible.

Proof Main idea:

$$B=0 \rightarrow K\left\{\frac{A}{A \vee B}\right\} \rightarrow \boxed{\begin{array}{c} \varphi \\ \dots \\ [x_i : VB \Rightarrow x_i]_A \check{A} \quad \begin{array}{c} \varphi \\ \dots \\ K\{A\} \end{array} \\ \parallel \\ [\check{A}^j \Rightarrow z_j]_B K\{AvB\} \\ \dots \\ \psi \end{array}}$$

eversion

$$\begin{array}{c} (A \dot{\beta} B) \times (C \dot{\beta} D) \\ \hline (A \dot{\alpha} C) \wedge (B \dot{\alpha} D) \end{array} \quad \begin{array}{l} \check{v} = \check{\lambda} = v \\ \hat{v} = \hat{\lambda} = \lambda \\ \check{\alpha} = \hat{\alpha} = \alpha \end{array}$$

(2) $\alpha, \beta \in \{V, \Lambda, \check{a}, b, c, \dots\}$

A	$A \vee 0$	A	$A \wedge I$
$A \vee 0$	A	$A \wedge I$	A

(1)	0	0	0 \otimes 0
0 \wedge 0	$\underline{0}$	$\underline{0}$	0
1 \vee 1	$\underline{1}$	$\underline{1}$	1
1	1	1	1 \otimes 1

TOTAL LINEARITY

$$\frac{(A \overset{\beta}{\dot{\beta}} B) \curlywedge (C \overset{\beta}{\dot{\beta}} D)}{(A \overset{\alpha}{\dot{\alpha}} C) \overset{\beta}{\dot{\beta}} (B \overset{\alpha}{\dot{\alpha}} D)}$$

$\check{V} = \check{\Lambda} = V$
 $\hat{V} = \hat{\Lambda} = \Lambda$
 $\check{\alpha} = \hat{\alpha} = \alpha$

$\alpha, \beta \in \{V, \Lambda, \check{a}, \check{b}, \check{c}, \dots\}$

TOTAL LINEARITY

Simple, natural proof theory.

$$\frac{(A \dot{\beta} B) \times (C \dot{\beta} D)}{(A \dot{\alpha} C) \wp (B \dot{\alpha} D)}$$

$\check{v} = \check{\lambda} = v$
 $\hat{v} = \hat{\lambda} = \lambda$
 $\check{\alpha} = \hat{\alpha} = \alpha$

$$\alpha, \beta \in \{v, \lambda, \check{a}, \hat{b}, c, \dots\}$$