Propositional proof systems and bounded arithmetic for logspace and nondeterministic logspace

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Virtual Seminar
Proof Society
proofsociety.org
October 7, 2020

includes joint work with Anupam Das and Alexander Knop

Setting:

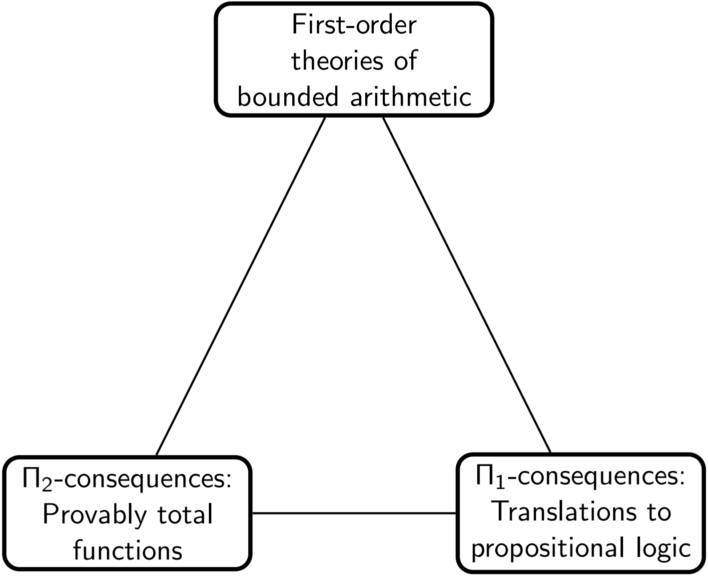
- Formal theories of weak fragments of Peano arithmetic
  - First- and second-order theories of bounded arithmetic
- $\forall\exists$ consequences: Provably total functions
  - Computational complexity characterizations
- $\forall$ consequences: Universal statements
  - Cook/Paris-Wilkie translation to propositional logic

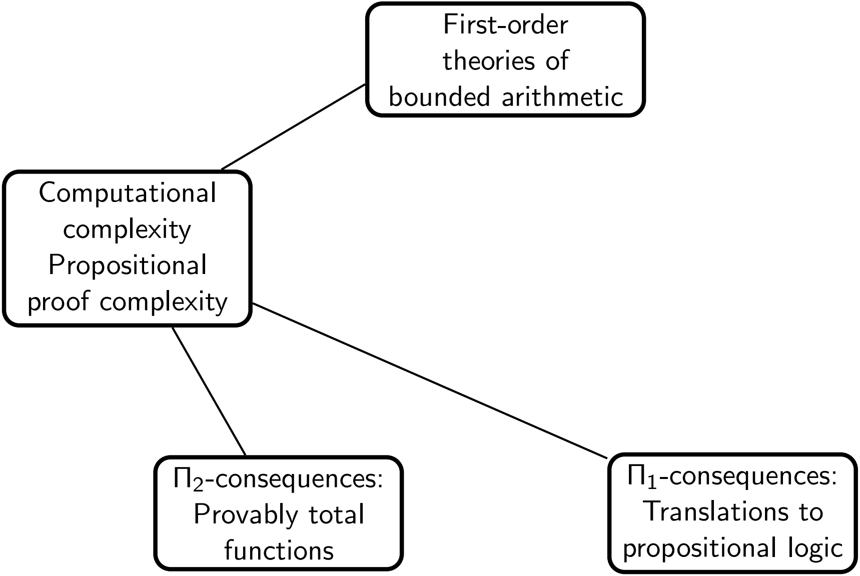
Underlying philosophy:

- A feasibly constructive proof that a function is total should provide a feasible method to compute it.
- A feasibly constructive proof of a universal statement should provide a feasible method to verify any given instance.

This talk (work-in-progress)

- Propositional and second-order systems for logspace and non-deterministic log space.


First-order theories of bounded arithmetic
Π_2-consequences: Provably total functions
Π_1-consequences: Translations to propositional logic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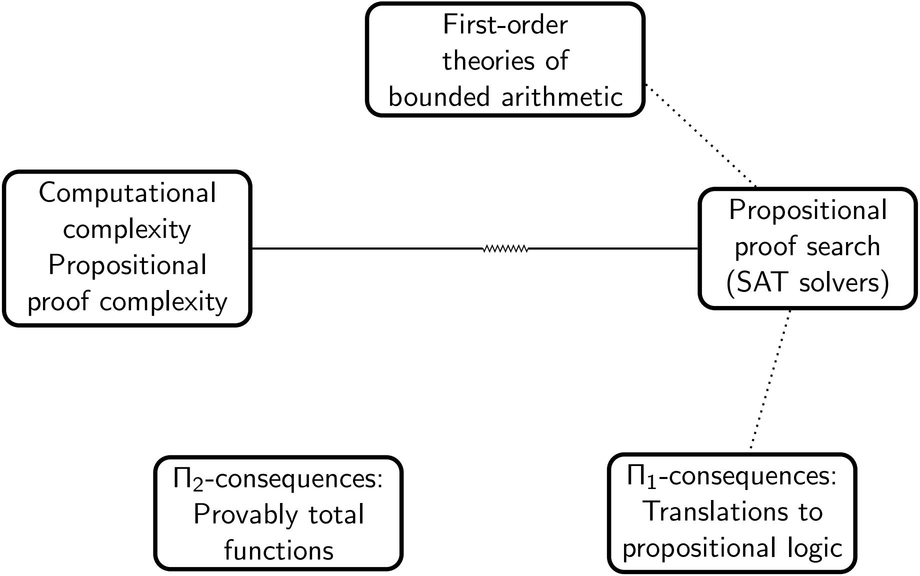
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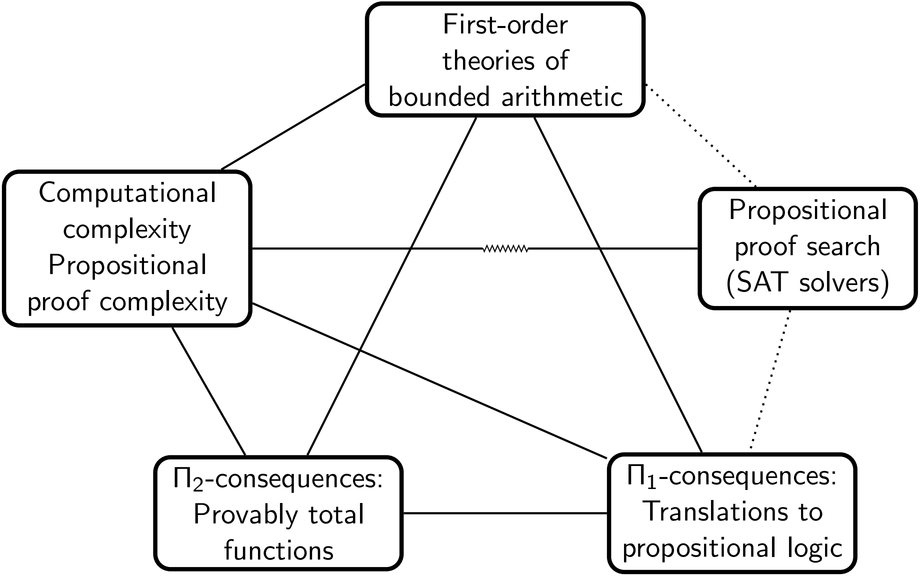
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First-order theory S^1_2 of arithmetic:

- Terms have polynomial growth rate (smash, $\#$, is used).
- Bounded quantifiers $\forall x \le t$, $\exists x \le t$.
- Sharply bounded quantifiers $\forall x \le |t|$, $\exists x \le |t|$,

  bound x by *log* (or *length*) of t.
- Classes Σ^{b}_i and Π^{b}_i of formulas are defined by counting bounded quantifiers, ignoring sharply bounded quantifiers.
- Σ^{b}_1 formulas express exactly the NP predicates.

  Σ^{b}_i, Π^{b}_i - express exactly the predicates at the i-th level of the polynomial time hierarchy.
- S^1_2 has *polynomial induction* PIND, equivalently *length induction* (LIND), for Σ^{b}_1 formulas A (i.e., NP formulas):

$$A(0) \land (\forall x)(A(x) \to A(x+1)) \to (\forall x)A(|x|)$$

(1) Provably total functions of S^1_2:

- The $\forall\Sigma^b_1$-definable functions (aka: *provably total functions*) are precisely the polynomial time computable functions.

(2) Translation to propositional logic ("Cook translation")

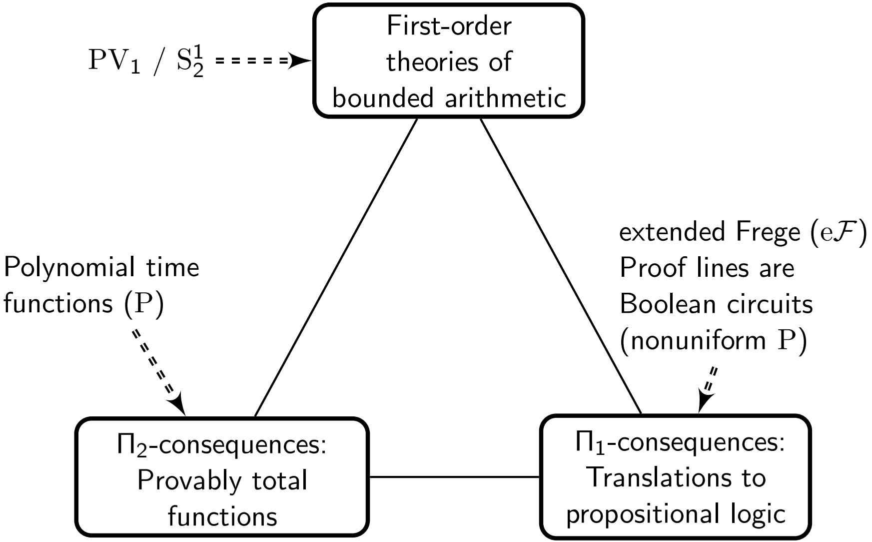
- Any polynomial identity ($\forall\Sigma^b_0$-property) provable in PV / S^1_2, has a natural translation to a family F of propositional formulas. These formulas have polynomial size extended Frege ($\mathrm{e}\mathcal{F}$) proofs.

(3) S^1_2 proves the consistency of $\mathrm{e}\mathcal{F}$. Conversely, any propositional proof system (p.p.s.) S^1_2 proves is consistent(provably) polynomially simulated by $\mathrm{e}\mathcal{F}$.

(4) Lines (formulas) in an $\mathrm{e}\mathcal{F}$ proof correspond to Boolean circuits. The circuit value problem is complete for P (polynomial time).

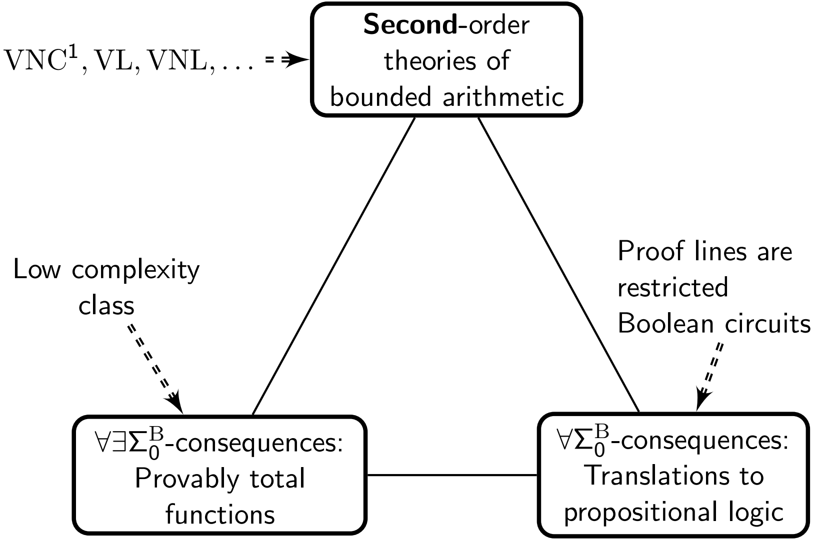
First-order theories work well for NC^2 and stronger classes.

E.g., for polynomial time:



For complexity classes below NC^2

[Clote-Takeuti; Zambella; Arai; Cook, Morioka, Perron, Kolokolova, Nguyen]



Weak second-order theories

These second-order theories use
(a) first-order objects playing the role of sharply bounded objects,
(b) second-order objects playing the role of inputs and outputs.
Base theory V^0 has comprehension and induction for bounded first-order formulas (with second order free variables).

Syntax:

- First-order bounded quantifiers $\forall x{\le}t, \exists x{\le}t$ - range over small objects, namely "integers".
- Second-order quantifiers $\forall X, \exists X$ - range over large objects, namely (finite) sets of integers.
- $x \in Y$ or $Y(x)$ - set membership.
- Σ_0^{B} formulas have only bounded first-order quantifiers and no second-order quantifiers
- First-order arithmetic operations: $0, S, \mathrm{pd}, +, \cdot, \le, =$
- $|X|$ - maximal element in X (optional).

Axioms of base theory V_0:

- "Basic" axioms of first-order functions and $\leq$ and $=$.
- Boundedness: $\exists y\, \forall x (A(x) \rightarrow x \leq y)$.
- Minimization: $A(b) \rightarrow \exists x\, [A(x) \wedge \forall y < x. \neg A(y)]$.
- Σ_0^{B}-Comprehension $\exists X\, \forall y \leq a\, [X(y) \leftrightarrow \varphi(y)]$,
  for φ a Σ_0^{B}-formula (with parameters allowed).

Theories VL (for logspace) and VNL (for nondeterministic logspace) have additional axioms for totality of L and NL complete functions — *on next slides ...*

A theory for L (log space) [Z, P, CN]

- VL - is V^0 plus the totality of log-bounded recursion.
- Provably total functions are precisely the log-space computable functions.
- Cook translation is a tree-like propositional proof system GL^* for Σ-$\mathrm{CNF}(2)$ formulas, a class of QBF formulas complete for log space. [J]

Log-bounded recursion axiom [Zambella]

$$
\begin{aligned}
&(\forall x{\le}a)(\exists y{\le}a)A(x,y) \to \exists X[X(0,0)\wedge \\
&\qquad (\forall i{\le}b)(\forall y{\le}a)(X(i,y) \to (\forall y'{<}y)\neg X(i,y')) \wedge \\
&\qquad (\forall i{<}b)(\exists y{\le}a)(\exists y'{\le}a)(X(i,y) \wedge X(i{+}1,y') \wedge A(y,y'))]
\end{aligned}
$$

Intuition: $A(x,y)$ denotes the (wlog deterministic) step for a path; defines a directed graph with out-degree ≥ 1..
$X(i,y)$ means y is the i-th vertex in the path.
The axiom asserts existence of a path of length b.
The predicate $X(i,y)$ is log-space complete.

A theory for $\mathbf{NL}$ (non-deterministic log space) [CK, P, CN]

- VNL - is V^0 plus the existence of a distance predicate for graph reachability.
- Provably total functions are precisely the polynomial growth rate functions with NL bit graph.
- Cook translation is a tree-like propositional proof system GNL^* for ΣKrom formulas, a class of QBF formulas complete for NL. [G]

Reachability/connectivity axiom [NC]

$$
\begin{aligned}
&(\exists X)[(\forall y{\le}a)(X(0,y) \leftrightarrow y=0) \wedge \\
&\qquad (\forall y{\le}a)(\forall i{<}b)[\,X(i{+}1,y) \leftrightarrow \\
&\qquad\qquad\qquad [\,X(i,y) \vee (\exists y'{<}a)(X(i,y') \wedge A(y',y))\,]\,]\,]
\end{aligned}
$$

Intuition: $A(x,y)$ denotes a possible step for a path.
$X(i,y)$ means y is reachable from 0 in $\le i$ steps.
The predicate $X(i,y)$ is NL complete.

Formal Theory	Propositional Proof System	Total Functions	
PV, S^1_2, VPV	$\mathrm{e}\mathcal{F}$, G^*_1	P	[C, B, CN]
T^1_2, S^2_2	G_1, G^*_2	$\le_{1\text{-}1}(\mathrm{PLS})$	[B, KP, KT, BK]
T^2_2, S^3_2	G_2, G^*_3	$\le_{1\text{-}1}(\mathrm{CPLS})$	[B, KP, KT, KST]
T^i_2, S^{i+1}_2	G_i, G^*_{i+1}	$\le_{1\text{-}1}(\mathrm{LLI}_i)$	[B, KP, KT, KNT]
PSA, U^1_2, W^1_1	QBF	*PSPACE*	[D, B, S]
V^1_2	**	*EXPTIME*	[B]
VNC^1	Frege ($\mathcal{F}$)	$\mathrm{ALogTime}$	[CT, A; CM, CN]
VL	GL^*	L	[Z, P, CN]
VNL	GNL^*	NL	[CK, P, CN]

PV, PSA - equational theories.
S^i_2, T^i_2 - first order
$\mathrm{U}^1_2, \mathrm{V}^1_2$, $\mathrm{VNC}^1, \mathrm{VL}, \mathrm{VNL}, \mathrm{VPV}$ - second order

Formal Theory	Propositional Proof System	Total Functions	
PV, S^1_2, VPV	$\mathrm{e}\mathcal{F}$, G^*_1	P	[C, B, CN]
T^1_2, S^2_2	G_1, G^*_2	$\leq_{1\text{-}1}(\mathrm{PLS})$	[B, KP, KT, BK]
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T^i_2, S^{i+1}_2	G_i, G^*_{i+1}	$\leq_{1\text{-}1}(\mathrm{LLI}_i)$	[B, KP, KT, KNT]
PSA, U^1_2, W^1_1	QBF	*PSPACE*	[D, B, S]
V^1_2	**	*EXPTIME*	[B]
VNC^1	Frege ($\mathcal{F}$)	$\mathrm{ALogTime}$	[CT, A; CM, CN]
VL	GL^*	L	[Z, P, CN]
VNL	GNL^*	NL	[CK, P, CN]

Using Cook translation to propositional proof systems (p.p.s.'s)

$\mathcal{F}, \mathrm{e}\mathcal{F}$ - Frege and extended Frege.

G_i, QBF - quantified propositional logics.

Starred (*) propositional systems are tree-like.

Formal Theory	Propositional Proof System	Total Functions	
PV, S_2^1, VPV	$\mathrm{e}\mathcal{F}$, G_1^*	P	[C, B, CN]
T_2^1, S_2^2	G_1, G_2^*	$\leq_{1\text{-}1}(\mathrm{PLS})$	[B, KP, KT, BK]
T_2^2, S_2^3	G_2, G_3^*	$\leq_{1\text{-}1}(\mathrm{CPLS})$	[B, KP, KT, KST]
T_2^i, S_2^{i+1}	G_i, G_{i+1}^*	$\leq_{1\text{-}1}(\mathrm{LLI}_i)$	[B, KP, KT, KNT]
PSA, U_2^1, W_1^1	QBF	*PSPACE*	[D, B, S]
V_2^1	**	*EXPTIME*	[B]
VNC^1	Frege ($\mathcal{F}$)	$\mathrm{ALogTime}$	[CT, A; CM, CN]
VL	GL^*	L	[Z, P, CN]
VNL	GNL^*	NL	[CK, P, CN]

PLS = Polynomial local search [JPY]

CPLS = "Colored" PLS [ST]

LLI = Linear local improvement

New Propositional Theories for L and NL [B-D-K]

The propositional theories GL^* and GNL^* are hard to work with since they are only indirectly connected with (non-uniform) log space and non-deterministic log space.

[Buss-Das-Knop'20]: Theories eLDT and eLNDT for propositional logic acting on

- Branching programs (eDT formulas), or
- Non-deterministic branching programs (eNDT formulas).

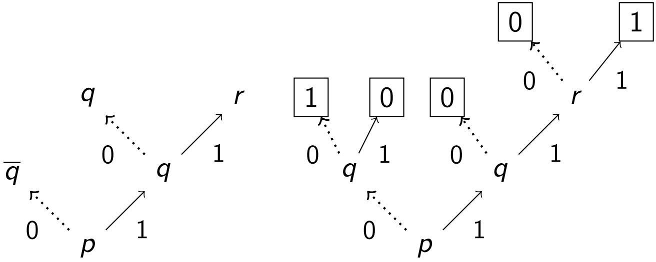
[Similar to a suggestion of Cook, but not using Prover-Liar games.]

Nomenclature: "DT" means "Decision Tree". An "eDT" ("extension Decision Tree") allows also extension variables, which converts the decision tree into a decision DAG, i.e., into a Branching Program (BP).

DT formulas

- **Atomic DT formulas:** p, $\overline{p}$, optionally 0 and 1.
- **Decision (or case/if-then-else) connective:** $(\varphi p \psi)$.
  Meaning "if p then ψ else φ" or "case(p,ψ,φ)".

Example:



These represent the equivalent formulas

$$\overline{q}\, p\, (q\, q\, r), \text{ and } (1q0)\, p\, (0\, q\, (0r1)).$$

Gentzen-style sequent calculus for DT/eDT formulas:

- Initial sequents, weak inferences, cut rule, and
- Decision connective rules:

$$\textit{dec-l:}\ \frac{A, \Gamma \longrightarrow \Delta, p \qquad p, B, \Gamma \longrightarrow \Delta}{(ApB), \Gamma \longrightarrow \Delta}$$

$$\textit{dec-r:}\ \frac{\Gamma \longrightarrow \Delta, A, p \qquad p, \Gamma \longrightarrow \Delta, B}{\Gamma \longrightarrow \Delta, (ApB)}$$

eDT - Extension DT formulas

Extension DT (eDT) formulas. Allow also introducing new extension variable, e, with a defining equation $e \leftrightarrow \varphi$.

eDT formulas, together with their defining equations express exactly (deterministic) branching programs.

Extension variables e can be used — unnegated — as atomic formulas, but cannot be used as decision literals.
The new initial sequents are $e \longrightarrow \varphi$ and $\varphi \longrightarrow e$.

- An eLDT proof is a sequence of sequents of eDT formulas inferred using the sequent calculus rules.
- The eDT formulas are used in conjunction with the defining equations for extension variables.
- Evaluation of eDT formulas is log-space complete. Thus, each line of an eLDT proof expresses a (non-uniform) logspace property.

Theorem ([BDK] - work in progress)

- *The* $\forall\Sigma_0^{\mathrm{B}}$*-consequences of* VL *have natural propositional translations which have polynomial size* eLDT*-proofs.*
- VL *can prove the consistency of* eLDT *proofs.*
- *Any propositional proof system which is* VL*-provably sound is p-simulated by* eLDT*.*

eLNDT for non-deterministic logspace

- NDT and eNDT formulas are defined exactly like DT and eDT formulas, but allowing also the connective $\vee$ (disjunction). Now it is important that extension variables can not be negated!
- An NDT formula expresses a "nondeterministic decision tree".
- An eNDT formula (together with defining equations for extension variables) expresses a "nondeterministic branching program".
- Evaluation of eNDT formulas (nondeterministic branching programs) is complete for non-deterministic logspace.
- An eLNDT proof consists of sequents of eNDT formulas.
- The permitted rules of inference are the inference rules for eLDT proofs (including the decision rule), plus the usual Gentzen $\vee$:left and $\vee$:right rules.

Theorem ([BDK] - work in progress)

- *The* $\forall\Sigma_0^{\mathrm{B}}$*-consequences of* VNL *have natural propositional translations which have polynomial size* eLNDT*-proofs.*
- VNL *can prove the consistency of* eLNDT *proofs.*
- *Any propositional proof system which is* VNL*-provably sound is p-simulated by* eLDT*.*

This includes (re)proving the Immerman-Szelepcsényi theorem that $\mathrm{NL} = \mathrm{coNL}$ in VNL. C.f. [CK, P].

Thank you for virtual attendance!