Axiomatizing provable *n*-provability

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Preliminaries: theories

We deal with arithmetical theories in the language $0, (\cdot)', +, \times, \exp$.

Arithmetical hierarchy:

- $\Sigma_0 = \Pi_0 = \Delta_0(\exp) =$ bounded arithmetical formulas,
- $\varphi \in \Pi_n \Rightarrow \exists x_1, \ldots, x_k\, \varphi \in \Sigma_{n+1}$,
- $\psi \in \Sigma_n \Rightarrow \forall x_1, \ldots, x_k\, \psi \in \Pi_{n+1}$.

Elementary arithmetic — Peano arithmetic

$$\mathrm{EA} \equiv \mathrm{I}\Delta_0(\exp) \subseteq \mathrm{EA}^+ \subseteq \mathrm{I}\Sigma_1 \subseteq \cdots \subseteq \mathrm{I}\Sigma_m \subseteq \cdots \subseteq \mathrm{PA} \equiv \bigcup_{m \in \omega} \mathrm{I}\Sigma_m.$$

$\mathrm{EA}^+ := \mathrm{EA} + \forall x, y\, \exists z\, 2^x_y = z$, where $2^x_0 := 2^x$ and $2^x_{y+1} := 2^{2^x_y}$.
$\mathrm{I}\Sigma_m := \mathrm{EA}$ + induction schema restricted to Σ_m-formulas.

$$T \subseteq U \Leftrightarrow \text{every theorem of } T \text{ is a theorem of } U$$
$$T \equiv U \Leftrightarrow T \text{ and } U \text{ have the same theorems}$$
$$T \equiv_{\Sigma_n} U \Leftrightarrow T \text{ and } U \text{ have the same } \Sigma_n\text{-consequences}$$

Preliminaries: provability

For every r.e. consistent T we fix some $\Delta_0(\exp)$-formula

$$\mathrm{Prf}_T(p, x) := p \text{ codes a proof of a formula } \varphi \text{ with } x = \ulcorner \varphi \urcorner.$$

The provability predicate for T is given by $\Box_T(x) := \exists p\, \mathrm{Prf}_T(p, x)$.

The classes Π_n have truth definitions $\mathrm{True}_{\Pi_n}(z) \in \Pi_n$ such that

$$\forall \varphi \in \Pi_n \quad \mathrm{EA} \vdash \mathrm{True}_{\Pi_n}(\ulcorner \varphi(\underline{x_1}, \ldots, \underline{x_k}) \urcorner) \leftrightarrow \varphi(x_1, \ldots, x_k).$$

φ is *n*-provable in T, if φ is provable in T + all true Π_n-sentences.

$$[n]_T \varphi := \exists \pi \left(\mathrm{True}_{\Pi_n}(\ulcorner \pi \urcorner) \wedge \Box_T(\ulcorner \pi \to \varphi \urcorner) \right).$$

$[n]_T$ satisfies the derivability conditions and is provably Σ_{n+1}-complete:

1. If $T \vdash \varphi$, then $\mathrm{EA} \vdash [n]_T \varphi$.
2. $\mathrm{EA} \vdash [n]_T(\varphi \to \psi) \to ([n]_T \varphi \to [n]_T \psi)$.
3. $\mathrm{EA} \vdash [n]_T \varphi \to [n]_T [n]_T \varphi$.
4. $\mathrm{EA} \vdash \sigma(\vec{x}) \to [n]_T \sigma(\underline{\vec{x}})$, whenever $\sigma(\vec{x})$ is a Σ_{n+1}-formula.

Provable n-provability

A formula φ is provably n-provable in PA, if $\mathrm{PA} \vdash [n]_{\mathrm{PA}}\varphi$.

$$\mathrm{PA} \vdash \Box_{\mathrm{PA}}\varphi \leftrightarrow \mathrm{PA} \vdash \varphi \Longrightarrow \text{provable 0-provability} \leftrightarrow \text{provability.}$$

Although for a fixed $n > 0$ the set of all formulas n-provable in PA is not r.e., the set of all provably n-provable formulas is an r.e. theory extending PA.

Problem. Find an explicit axiomatization of the theory

$$\{\varphi \mid \mathrm{PA} \vdash [n]_{\mathrm{PA}}\varphi\} \text{ for a fixed } n > 0.$$

More generally, given a pair T and S of r.e. consistent extensions of EA we consider the set of formulas T-provably n-provable in S

$$C^n_S(T) := \{\varphi \mid T \vdash [n]_S\varphi\}.$$

$C^n_S(T)$ is an r.e. extension of S with the provability predicate $\Box_T[n]_S$.

Main question. How can we axiomatize $C^n_S(T)$ in terms of T and S?

Iterated local reflection principles

The answer will be given using iterated local reflection principles.

The local reflection principles are the following schemata

- full schema $\mathrm{Rfn}(T) := \{\Box_T \varphi \to \varphi \mid \varphi \text{ is a sentence}\}$,
- partial schema $\mathrm{Rfn}_{\Sigma_n}(T) := \{\Box_T \varphi \to \varphi \mid \varphi \text{ is a } \Sigma_n\text{-sentence}\}$.

The relativized versions $\mathrm{Rfn}^n(T)$ and $\mathrm{Rfn}^n_{\Sigma_n}(T)$ of the above principles are obtained by replacing $\Box_T$ with $[n]_T$.

Turing considered the transfinite iterations of such principles along recursive ordinals $(D, \prec)$ (Turing progressions, 1939).

$$\mathrm{Rfn}(T)_0 := T, \quad \mathrm{Rfn}(T)_{\alpha+1} := T + \mathrm{Rfn}(\mathrm{Rfn}(T)_\alpha),$$

$$\mathrm{Rfn}(T)_\lambda := \bigcup_{\beta<\lambda} \mathrm{Rfn}(T)_\beta, \quad \lambda \in \mathrm{Lim}.$$

Formally, the sequence of theories $\mathrm{Rfn}(T)_\alpha, \alpha \in D$ is defined via the fixed-point lemma applied to the formalization of the following

$$\mathrm{Rfn}(T)_\alpha \equiv T + \{\mathrm{Rfn}(\mathrm{Rfn}(T)_\beta) \mid \beta \prec \alpha\}.$$

Local reflection is provably 1-provable

Fact. $\mathrm{Rfn}(S) \subseteq C^1_S(T)$.

Proof. Consider an arbitrary instance $\Box_S \varphi \to \varphi$ of $\mathrm{Rfn}(S)$. We have

$$\begin{aligned} T \vdash \Box_S \varphi &\to [1]_S \varphi \\ &\to [1]_S(\Box_S \varphi \to \varphi). \end{aligned}$$

The sentence $\neg\Box_S \varphi$ is Π_1, so by provable Σ_2-completeness of $[1]_S$

$$\begin{aligned} T \vdash \neg\Box_S \varphi &\to [1]_S\,(\neg\Box_S \varphi) \\ &\to [1]_S(\Box_S \varphi \to \varphi), \end{aligned}$$

This shows $T \vdash [1]_S(\Box_S \varphi \to \varphi)$, whence $\mathrm{Rfn}(S) \subseteq C^1_S(T)$.

Iterated reflection is provably 1-provable

In the same fashion using transfinite induction in PA one can show

$$\mathrm{Rfn}(S)_{\omega_m} \subseteq C^1_S(\mathrm{I}\Sigma_m) \text{ and } \mathrm{Rfn}(S)_{\varepsilon_0} \subseteq C^1_S(\mathrm{PA}).$$

where $\omega_0 := 1, \omega_{m+1} := \omega^{\omega_m}$ and $\varepsilon_0 := \sup\{\omega_m \mid m \in \omega\}$.

Natural conjecture: $C^1_S(\mathrm{I}\Sigma_m) \equiv \mathrm{Rfn}(S)_{\omega_m}$ and $C^1_S(\mathrm{PA}) \equiv \mathrm{Rfn}(S)_{\varepsilon_0}$.

The main difficulty is to prove the reverse inclusions

$$C^1_S(\mathrm{I}\Sigma_m) \subseteq \mathrm{Rfn}(S)_{\omega_m} \text{ and } C^1_S(\mathrm{PA}) \subseteq \mathrm{Rfn}(S)_{\varepsilon_0}.$$

Main results: $n = 1$

We obtain the following results on provable 1-provability:

- For any $m \geqslant 0$ we have $C^1_S(\mathrm{I}\Sigma_m) \equiv \mathrm{Rfn}(S)_{\omega_m}$.
- It follows that $C^1_S(\mathrm{PA}) \equiv \mathrm{Rfn}(S)_{\varepsilon_0}$.
- In general, if α is the Σ^0_2-ordinal of a (Σ^0_2-regular) theory T measured w.r.t. the transfinite iterations of local Σ_2-reflection schema over EA, i.e., α is the least ordinal in $(D, \prec)$ such that

$$T \equiv_{\Sigma_2} \mathrm{Rfn}_{\Sigma_2}(\mathrm{EA})_\alpha,$$

  then we have $C^1_S(T) \equiv \mathrm{Rfn}(S)_{1+\alpha}$.
- All these equivalences are provable in EA^+.
- $C^1_{\mathrm{EA}}(\mathrm{I}\Sigma_m)$ has superexponential speed-up over $\mathrm{Rfn}(\mathrm{EA})_{\omega_m}$, i.e., an $\mathrm{I}\Sigma_m$-proof of 1-provability of φ can be "much shorter" than the shortest proof of φ from iterated reflection.

Main results: $n > 1$

The results above can be generalized to the case of provable n-provability for $n > 1$. In particular,

- For any $m \geqslant n > 0$ we have $C_S^{n+1}(\mathrm{I}\Sigma_m) \equiv \mathrm{Rfn}^n(S)_{\omega_{m-n}}$.
- It follows that $C_S^{n+1}(\mathrm{PA}) \equiv \mathrm{Rfn}^n(S)_{\varepsilon_0}$.
- If α is the Σ^0_{n+2}-ordinal of a (Σ^0_{n+2}-regular) theory T measured w.r.t. the transfinite iterations of $\mathrm{Rfn}^n_{\Sigma_{n+2}}(\mathrm{EA})$, then

$$C_S^{n+1}(T) \equiv \mathrm{Rfn}^n(S)_{1+\alpha}.$$

- In case $\mathrm{I}\Sigma_n \subseteq S$ these equivalences are provable in EA^+.

Σ_2-conservation results

We want to show $C^1_S(\mathrm{I}\Sigma_m) \subseteq \mathrm{Rfn}(S)_{\omega_n} \Longrightarrow$ it would be more convenient if we could replace $\mathrm{I}\Sigma_m$ under $C^1_S(\cdot)$ with an equivalent form of iterated reflection.

$[1]_S\varphi$ is a Σ_2-sentence $\Rightarrow C^1_S(T)$ depends on Σ_2-consequences of T.

Theorem (Beklemishev, Visser; 2005). Σ_2-consequences of $\mathrm{I}\Sigma_m$ are axiomatized by $\mathrm{Rfn}_{\Sigma_2}(\mathrm{EA})_{\omega_m}$ for $m > 0$.

Theorem (Beklemishev; 2003). $\mathrm{EA} \vdash \forall\alpha\, \mathrm{Rfn}_{\Sigma_2}(T)_\alpha \equiv_{\Sigma_2} \mathrm{Rfn}(T)_\alpha$.

Lemma. $C^1_S(\mathrm{I}\Sigma_m) \equiv C^1_S(\mathrm{Rfn}_{\Sigma_2}(\mathrm{EA})_{\omega_m}) \equiv C^1_S(\mathrm{Rfn}(\mathrm{EA})_{\omega_m})$.

$C^1_S(\cdot)$ permutes with $\mathrm{Rfn}(\cdot)_\alpha$

Lemma. $\mathsf{EA} \vdash \forall\alpha\, C^1_S(\mathrm{Rfn}(T)_\alpha) \equiv \mathrm{Rfn}(C^1_S(T))_\alpha$.

Proof (idea). Argue in EA by reflexive induction on α:

$$\frac{\forall\alpha(\Box_{\mathsf{EA}}(\forall\beta \prec \underline{\alpha}\, A(\beta)) \to A(\alpha))}{\forall\alpha\, A(\alpha)}$$

Characterizing $C^1_S(\mathsf{EA})$

Using the permutation property we get

$$C^1_S(\mathsf{I}\Sigma_m) \equiv C^1_S(\mathsf{Rfn}(\mathsf{EA})_{\omega_m}) \equiv \mathsf{Rfn}(C^1_S(\mathsf{EA}))_{\omega_m},$$

hence it is only left to find $C^1_S(\mathsf{EA})$.

Lemma. $\mathsf{EA}^+ \vdash C^1_S(\mathsf{EA}) \equiv S + \mathsf{Rfn}(S)$.

Proof (idea). $\supseteq$ was shown above.

For $\subseteq$ we apply a version of the Herbrand theorem for Σ_2-formulas.

Proof: a general outline (for $n = 1$)

Combining all the Lemmas we get the following chain of equivalences

	$C^1_S(\mathrm{I}\Sigma_m)$
Σ_2-consequences of $\mathrm{I}\Sigma_m$	$\equiv$
	$C^1_S(\mathrm{Rfn}_{\Sigma_2}(\mathrm{EA})_{\omega_m})$
Σ_2-conservativity of Rfn over Rfn_{Σ_2}	$\equiv$
	$C^1_S(\mathrm{Rfn}(\mathrm{EA})_{\omega_m})$
$C^1_S(\cdot)$ permutes with $\mathrm{Rfn}(\cdot)_\alpha$	$\equiv$
	$\mathrm{Rfn}(C^1_S(\mathrm{EA}))_{\omega_m}$
$C^1_S(\mathrm{EA}) \equiv S + \mathrm{Rfn}(S)$	$\equiv$
	$\mathrm{Rfn}(S + \mathrm{Rfn}(S))_{\omega_m}$
$1 + \omega_m = \omega_m$ for $m > 0$	$\equiv$
	$\mathrm{Rfn}(S)_{\omega_m}$

Relativization to the case $n > 1$

The same strategy applies to the case $n > 1$.

Theorem (Beklemishev, Visser; 2005). Σ_{n+2}-consequences of $\mathrm{I}\Sigma_m$ are axiomatized by $\mathrm{Rfn}^n_{\Sigma_{n+2}}(\mathrm{EA})_{\omega_{m-n}}$ for $m \geqslant n > 0$.

Theorem (Beklemishev; 2003). $\mathrm{EA} \vdash \forall\alpha\, \mathrm{Rfn}^n_{\Sigma_{n+2}}(T)_\alpha \equiv_{\Sigma_{n+2}} \mathrm{Rfn}^n(T)_\alpha$.

Lemma. $\mathrm{EA} \vdash \forall\alpha\, C^{n+1}_S(\mathrm{Rfn}(T)_\alpha) \equiv \mathrm{Rfn}^n(C^{n+1}_S(T))_\alpha$.

Lemma. $C^{n+1}_S(\mathrm{EA}) \equiv S + \mathrm{Rfn}^n(S)$. Moreover, if $\mathrm{I}\Sigma_n \subseteq S$ then this equivalence is provable in EA^+.

Proof (idea). Relativization of the syntactic proof for the case $n = 1$ (via the Herbrand theorem) uses Σ_n-induction, whence the requirement on S appears.

To prove the equivalence for arbitrary $S \supseteq \mathrm{EA}$ we argue model-theoretically using the substructure $K^{n+1}(M) \prec_{\Sigma_{n+1}} M$ of Σ_{n+1}-definable elements and the equivalence $\mathrm{Rfn}^n_{\Sigma_{n+1}}(\mathrm{EA}) \equiv \mathrm{I}\Sigma^-_n$.

L. D. Beklemishev and A. Visser. On the limit existence principles in elementary arithmetic and Σ_n^0-consequences of theories. *Annals of Pure and Applied Logic*, 136(1–2):56–74, 2005.

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Thank you for your attention!