Axiomatizing provable n-provability

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Preliminaries: theories

We deal with arithmetical theories in the language $0, (\cdot)', +, \times, \exp$.

Arithmetical hierarchy:

- $\Sigma_0 = \Pi_0 = \Delta_0(\exp) =$ bounded arithmetical formulas,
- $\varphi \in \Pi_n \Rightarrow \exists x_1, \dots, x_k\, \varphi \in \Sigma_{n+1}$ and $\psi \in \Sigma_n \Rightarrow \forall x_1, \dots, x_k\, \psi \in \Pi_{n+1}$.

Elementary arithmetic (base theory)   **Peano arithmetic**

$$\mathsf{EA} \equiv \mathsf{I}\Delta_0(\exp) \subseteq \mathsf{EA}^+ \subseteq \mathsf{I}\Sigma_1 \subseteq \dots \subseteq \mathsf{I}\Sigma_m \subseteq \dots \subseteq \mathsf{PA} \equiv \bigcup_{m<\omega} \mathsf{I}\Sigma_m.$$

$\mathsf{EA}^+ := \mathsf{EA} + \forall x, y\, \exists z\, 2^x_y = z$, where $2^x_0 := 2^x$ and $2^x_{y+1} := 2^{2^x_y}$.

$\mathsf{I}\Sigma_m := \mathsf{EA}$ + induction schema restricted to Σ_m-formulas with parameters.

$$T \subseteq U \Leftrightarrow \text{every theorem of } T \text{ is a theorem of } U$$
$$T \equiv U \Leftrightarrow T \text{ and } U \text{ have the same theorems}$$
$$T \equiv_{\Sigma_n} U \Leftrightarrow T \text{ and } U \text{ have the same } \Sigma_n\text{-theorems}$$

Preliminaries: provability

For every r.e. consistent T we fix some $\Delta_0(\exp)$-formula

$$\mathrm{Prf}_T(p, x) := p \text{ codes a proof of the formula } \varphi \text{ with } x = \ulcorner \varphi \urcorner.$$

The **provability predicate** for T is given by $\Box_T(x) := \exists p\, \mathrm{Prf}_T(p, x)$.

φ is n**-provable** in T, if φ is provable in T together with all true Π_n-sentences.

The classes Π_n for $n > 0$ have **truth definitions** $\mathrm{True}_{\Pi_n}(z) \in \Pi_n$ such that

$$\forall \varphi \in \Pi_n \quad \mathsf{EA} \vdash \mathrm{True}_{\Pi_n}(\ulcorner \varphi(\underline{x_1}, \dots, \underline{x_k}) \urcorner) \leftrightarrow \varphi(x_1, \dots, x_k),$$

which allow to formalize the notion of n-provability as follows

$$[n]_T \varphi := \exists \pi\, (\mathrm{True}_{\Pi_n}(\ulcorner \pi \urcorner) \wedge \Box_T(\ulcorner \pi \to \varphi \urcorner)).$$

$[n]_T$ satisfies the **derivability conditions** and is **provably** Σ_{n+1}**-complete**:

1. If $T \vdash \varphi$, then $\mathsf{EA} \vdash [n]_T \varphi$.
2. $\mathsf{EA} \vdash [n]_T(\varphi \to \psi) \to ([n]_T \varphi \to [n]_T \psi)$.
3. $\mathsf{EA} \vdash [n]_T \varphi \to [n]_T [n]_T \varphi$.
4. $\mathsf{EA} \vdash \sigma(\vec{x}) \to [n]_T \sigma(\vec{x})$, whenever $\sigma(\vec{x})$ is a Σ_{n+1}-formula.

Provable n-provability

A formula φ is **provably** n**-provable** in PA, if $\mathsf{PA} \vdash [n]_{\mathsf{PA}} \varphi$.

$$\mathsf{PA} \vdash \Box_{\mathsf{PA}} \varphi \leftrightarrow \mathsf{PA} \vdash \varphi \Longrightarrow \text{provable 0-provability} \leftrightarrow \text{provability}.$$

Although for a fixed $n > 0$ the set of all formulas n-provable in PA is not r.e., the set of all **provably** n-provable formulas is an r.e. theory extending PA.

Problem. Find an explicit axiomatization of the theory

$$\{\varphi \mid \mathsf{PA} \vdash [n]_{\mathsf{PA}} \varphi\} \text{ for a fixed } n > 0.$$

More generally, given a pair T and S of r.e. consistent extensions of EA we consider the set of formulas **T-provably n-provable in S**

$$C^n_S(T) := \{\varphi \mid T \vdash [n]_S \varphi\}.$$

$C^n_S(T)$ is an r.e. theory extending S, which can be viewed as a theory with the provability predicate $\Box_T [n]_S$.

Main question. How can we axiomatize $C^n_S(T)$ in terms of T and S?

Iterated local reflection principles

The answer will be given in terms of **iterated local reflection principles**.

The **local reflection principles** are the following schemata

- full local reflection $\mathrm{Rfn}(T) := \{\Box_T \varphi \to \varphi \mid \varphi \text{ is a sentence}\}$,
- partial local reflection $\mathrm{Rfn}_{\Sigma_n}(T) := \{\Box_T \varphi \to \varphi \mid \varphi \text{ is a } \Sigma_n\text{-sentence}\}$.

The relativized versions $\mathrm{Rfn}^n(T)$ and $\mathrm{Rfn}^n_{\Sigma_n}(T)$ of the above principles are obtained by replacing $\Box_T$ with $[n]_T$.

Turing considered the **transfinite iterations** of such principles along recursive ordinals $(D, \prec)$ (**Turing progressions**, 1939).

$$\mathrm{Rfn}(T)_0 := T,$$
$$\mathrm{Rfn}(T)_{\alpha+1} := T + \mathrm{Rfn}(\mathrm{Rfn}(T)_\alpha),$$
$$\mathrm{Rfn}(T)_\lambda := \bigcup_{\beta<\lambda} \mathrm{Rfn}(T)_\beta, \quad \lambda \in \mathrm{Lim}.$$

Formally, the sequence of theories $\mathrm{Rfn}(T)_\alpha, \alpha \in D$ is defined via the fixed-point lemma applied to the formalization of the following equivalence in EA

$$\mathrm{Rfn}(T)_\alpha \equiv T + \{\mathrm{Rfn}(\mathrm{Rfn}(T)_\beta) \mid \beta \prec \alpha\}.$$

Local reflection is provably 1-provable

Fact. $\mathrm{Rfn}(S) \subseteq C^1_S(T)$.

Using transfinite induction in PA one can generalize this fact and show

$$\mathrm{Rfn}(S)_{\omega_m} \subseteq C^1_S(\mathsf{I}\Sigma_m) \text{ and } \mathrm{Rfn}(S)_{\varepsilon_0} \subseteq C^1_S(\mathsf{PA}).$$

where $\omega_0 := 1, \omega_{m+1} := \omega^{\omega_m}$ and $\varepsilon_0 := \sup\{\omega_m \mid m \in \omega\}$.

Natural hypothesis: $C^1_S(\mathsf{I}\Sigma_m) \equiv \mathrm{Rfn}(S)_{\omega_m}$ and $C^1_S(\mathsf{PA}) \equiv \mathrm{Rfn}(S)_{\varepsilon_0}$.

The main difficulty is to prove the **reverse inclusions**.

Main results: provable 1-provability

We obtain the following **results on provable 1-provability:**

- For any $m \geqslant 0$ we have $C^1_S(\mathsf{I}\Sigma_m) \equiv \mathrm{Rfn}(S)_{\omega_m}$.
- It follows that $C^1_S(\mathsf{PA}) \equiv \mathrm{Rfn}(S)_{\varepsilon_0}$.
- In general, if α is the **Σ^0_2-ordinal** of a (Σ^0_2-regular) theory T measured w.r.t. the transfinite iterations of local Σ_2-reflection schema over EA, i.e., α is the least ordinal in $(D, \prec)$ such that $T \equiv_{\Sigma_2} \mathrm{Rfn}_{\Sigma_2}(\mathsf{EA})_\alpha$, then we have
$$C^1_S(T) \equiv \mathrm{Rfn}(S)_{1+\alpha}.$$
- All these equivalences are provable in EA^+.
- $C^1_{\mathsf{EA}}(\mathsf{I}\Sigma_m)$ has superexponential speed-up over $\mathrm{Rfn}(\mathsf{EA})_{\omega_m}$.

Main results: provable n-provability for $n > 1$

For $n > 1$ we get the following generalizations of the results above

- For any $m \geqslant n > 0$ we have $C^{n+1}_S(\mathsf{I}\Sigma_m) \equiv \mathrm{Rfn}^n(S)_{\omega_{m-n}}$.
- It follows that $C^{n+1}_S(\mathsf{PA}) \equiv \mathrm{Rfn}^n(S)_{\varepsilon_0}$.
- If α is the Σ^0_{n+2}-ordinal of a (Σ^0_{n+2}-regular) theory T measured w.r.t. the transfinite iterations of $\mathrm{Rfn}^n_{\Sigma_{n+2}}(\mathsf{EA})$, then
$$C^{n+1}_S(T) \equiv \mathrm{Rfn}^n(S)_{1+\alpha}.$$
- In case $\mathsf{I}\Sigma_n \subseteq S$ these equivalences are provable in EA^+.

Proof: a general outline (for $n = 1$)

The proof consists of **three key steps**.

1. Applying Σ_2-conservativity results

We want to show $C^1_S(\mathsf{I}\Sigma_m) \subseteq \mathrm{Rfn}(S)_{\omega_n} \Longrightarrow$ it would be more convenient, if we could replace $\mathsf{I}\Sigma_m$ under $C^1_S(\cdot)$ with an equivalent form of **iterated reflection**.

$$[1]_S \varphi \text{ is a } \Sigma_2\text{-sentence} \Rightarrow C^1_S(T) \text{ depends on the } \Sigma_2\textbf{-consequences} \text{ of } T$$

Theorem 1 (Beklemishev, Visser; [1]). For $m > 0$ provably in EA^+,

$$\mathsf{I}\Sigma_m \equiv_{\Sigma_2} \mathrm{Rfn}_{\Sigma_2}(\mathsf{EA})_{\omega_m}.$$

Theorem 2 (Beklemishev; [2]). Provably in EA,

$$\forall \alpha\, \mathrm{Rfn}_{\Sigma_2}(T)_\alpha \equiv_{\Sigma_2} \mathrm{Rfn}(T)_\alpha.$$

2. Permuting $C^1_S(\cdot)$ with $\mathrm{Rfn}(\cdot)_\alpha$

Lemma 3. Provably in EA, $\forall \alpha\, C^1_S(\mathrm{Rfn}(T)_\alpha) \equiv \mathrm{Rfn}(C^1_S(T))_\alpha$.
Proof (idea) goes by reflexive induction on α in EA.

3. Characterizing $C^1_S(\mathsf{EA})$

Lemma 4. Provably in EA^+, $C^1_S(\mathsf{EA}) \equiv S + \mathrm{Rfn}(S)$.
Proof (idea) uses the Herbrand theorem for Σ_2-formulas.

Combining all the results above we get the following **chain of equivalences**

$$C^1_S(\mathsf{I}\Sigma_m) \equiv C^1_S(\mathrm{Rfn}_{\Sigma_2}(\mathsf{EA})_{\omega_m}) \equiv C^1_S(\mathrm{Rfn}(\mathsf{EA})_{\omega_m}) \equiv \mathrm{Rfn}(C^1_S(\mathsf{EA}))_{\omega_m} \equiv \mathrm{Rfn}(S)_{\omega_m}$$

The same strategy applies to the case $n > 1$, since all the results above can be relativized. The only subtle point is that the syntactic proof of the relativized version of Lemma 4 seems to require $S \supseteq \mathsf{I}\Sigma_n$.

To prove the equivalence for arbitrary $S \supseteq \mathsf{EA}$ we argue model-theoretically using the **substructure** $K^{n+1}(M) \prec_{\Sigma_{n+1}} M$ **of Σ_{n+1}-definable elements**.

References

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