Ordinals and Hierarchies

Andreas Weiermann

Department of Mathematics
Ghent University

Proof Society Summer School

Ghent, 2-5 September 2018

OUTLINE

Outline

Outline

OUTLINE

OUTLINE

The class of ordinals is denoted by On and it satisfies the following postulates (which can be proved in *ZFC*):

The class of ordinals is denoted by On and it satisfies the following postulates (which can be proved in *ZFC*):

1. $(\mathrm{On}, <)$ is linearly ordered (= totally ordered).

The class of ordinals is denoted by On and it satisfies the following postulates (which can be proved in *ZFC*):

1. $(\mathrm{On}, <)$ is linearly ordered (= totally ordered).
2. if $\emptyset \subsetneq C \subseteq \mathrm{On}$ then C has a minimal element, denoted by $\min C$.

The class of ordinals is denoted by On and it satisfies the following postulates (which can be proved in *ZFC*):

1. $(\mathrm{On}, <)$ is linearly ordered (= totally ordered).
2. if $\emptyset \subsetneq C \subseteq \mathrm{On}$ then C has a minimal element, denoted by $\min C$.
3. the class $\{\xi \in \mathrm{On} \mid \xi < \alpha\}$ is a set for all $\alpha \in \mathrm{On}$.

The class of ordinals is denoted by On and it satisfies the following postulates (which can be proved in *ZFC*):

1. $(\mathrm{On}, <)$ is linearly ordered (= totally ordered).
2. if $\emptyset \subsetneq C \subseteq \mathrm{On}$ then C has a minimal element, denoted by $\min C$.
3. the class $\{\xi \in \mathrm{On} \mid \xi < \alpha\}$ is a set for all $\alpha \in \mathrm{On}$.
4. for every set $A \subseteq \mathrm{On}$ exists a $\gamma \in \mathrm{On}$ such that $\alpha < \gamma$ for all $\alpha \in A$.

UNIVERSITEIT GENT

The class of ordinals is denoted by On and it satisfies the following postulates (which can be proved in *ZFC*):

1. $(\mathrm{On}, <)$ is linearly ordered (= totally ordered).
2. if $\emptyset \subsetneq C \subseteq \mathrm{On}$ then C has a minimal element, denoted by $\min C$.
3. the class $\{\xi \in \mathrm{On} \mid \xi < \alpha\}$ is a set for all $\alpha \in \mathrm{On}$.
4. for every set $A \subseteq \mathrm{On}$ exists a $\gamma \in \mathrm{On}$ such that $\alpha < \gamma$ for all $\alpha \in A$.

We use the following abbreviations: $0 = \min \mathrm{On}$

The class of ordinals is denoted by On and it satisfies the following postulates (which can be proved in *ZFC*):

1. $(\mathrm{On}, <)$ is linearly ordered (= totally ordered).
2. if $\emptyset \subsetneq C \subseteq \mathrm{On}$ then C has a minimal element, denoted by $\min C$.
3. the class $\{\xi \in \mathrm{On} \mid \xi < \alpha\}$ is a set for all $\alpha \in \mathrm{On}$.
4. for every set $A \subseteq \mathrm{On}$ exists a $\gamma \in \mathrm{On}$ such that $\alpha < \gamma$ for all $\alpha \in A$.

We use the following abbreviations: $0 = \min \mathrm{On}$ and $\alpha' = \min\{\xi \in \mathrm{On} \mid \alpha < \xi\}$.

TRANSFINITE INDUCTION

Theorem

Let ϕ be a property of ordinals.

TRANSFINITE INDUCTION

Theorem

Let ϕ be a property of ordinals. If $\forall\alpha \in \mathbf{On} : (\forall\xi < \alpha : \phi(\xi)) \to \phi(\alpha)$ then $\forall\alpha \in \mathbf{On} : \phi(\alpha)$.

Transfinite induction

Theorem

Let ϕ be a property of ordinals. If $\forall\alpha \in \mathbf{On} : (\forall\xi < \alpha : \phi(\xi)) \to \phi(\alpha)$ then $\forall\alpha \in \mathbf{On} : \phi(\alpha)$.

Proof.

Assume the antecedence and assume that there exists a $\beta \in \mathbf{On}$ such that $\neg\phi(\beta)$.

TRANSFINITE INDUCTION

Theorem

Let ϕ be a property of ordinals. If $\forall\alpha \in \mathrm{On} : (\forall\xi < \alpha : \phi(\xi)) \to \phi(\alpha)$ then $\forall\alpha \in \mathrm{On} : \phi(\alpha)$.

Proof.
Assume the antecedence and assume that there exists a $\beta \in \mathrm{On}$ such that $\neg\phi(\beta)$. Define

$$C = \{\xi \in \mathrm{On} \mid \neg\phi(\xi)\}.$$

This class is non empty since $\beta \in C$.

UNIVERSITEIT GENT

TRANSFINITE INDUCTION

Theorem

Let ϕ be a property of ordinals. If $\forall\alpha \in \mathrm{On} : (\forall\xi < \alpha : \phi(\xi)) \to \phi(\alpha)$ then $\forall\alpha \in \mathrm{On} : \phi(\alpha)$.

Proof.

Assume the antecedence and assume that there exists a $\beta \in \mathrm{On}$ such that $\neg\phi(\beta)$. Define

$$C = \{\xi \in \mathrm{On} \mid \neg\phi(\xi)\}.$$

This class is non empty since $\beta \in C$. Put $\alpha_0 = \min C$.

TRANSFINITE INDUCTION

Theorem

Let ϕ be a property of ordinals. If $\forall\alpha \in \mathrm{On} : (\forall\xi < \alpha : \phi(\xi)) \to \phi(\alpha)$ then $\forall\alpha \in \mathrm{On} : \phi(\alpha)$.

Proof.

Assume the antecedence and assume that there exists a $\beta \in \mathrm{On}$ such that $\neg\phi(\beta)$. Define

$$C = \{\xi \in \mathrm{On} \mid \neg\phi(\xi)\}.$$

This class is non empty since $\beta \in C$. Put $\alpha_0 = \min C$. Then $\neg\phi(\alpha_0)$. By assumption there exists an $\alpha_1 < \alpha_0$ such that $\neg\phi(\alpha_1)$ since otherwise $\phi(\alpha_0)$. Contradicton with the minimality of α_0.

Limit ordinals

The class of limit ordinals (denoted by *Lim*) is defined by

$$\alpha \in \mathrm{Lim} \iff \alpha \neq 0 \wedge \forall \xi < \alpha : \xi' < \alpha$$

The least limit ordinal is $\omega = \min \mathrm{Lim}$.
For a set $A \subseteq \mathrm{On}$ define

$$\sup A = \min\{\xi \in \mathrm{On} \mid \forall \alpha \in A : \alpha \leq \xi\}$$

In particular we have $\sup \emptyset = 0$.

Limit ordinals

The class of limit ordinals (denoted by *Lim*) is defined by

$$\alpha \in \mathrm{Lim} \iff \alpha \neq 0 \wedge \forall \xi < \alpha : \xi' < \alpha$$

The least limit ordinal is $\omega = \min \mathrm{Lim}$.

LIMIT ORDINALS

The class of limit ordinals (denoted by *Lim*) is defined by

$$\alpha \in \mathrm{Lim} \iff \alpha \neq 0 \wedge \forall \xi < \alpha : \xi' < \alpha$$

The least limit ordinal is $\omega = \min \mathrm{Lim}$.
For a set $A \subseteq \mathrm{On}$ define

$$\sup A = \min\{\xi \in \mathrm{On} \mid \forall \alpha \in A : \alpha \leq \xi\}$$

LIMIT ORDINALS

The class of limit ordinals (denoted by *Lim*) is defined by

$$\alpha \in \mathrm{Lim} \iff \alpha \neq 0 \wedge \forall \xi < \alpha : \xi' < \alpha$$

The least limit ordinal is $\omega = \min \mathrm{Lim}$.
For a set $A \subseteq \mathrm{On}$ define

$$\sup A = \min\{\xi \in \mathrm{On} \mid \forall \alpha \in A : \alpha \leq \xi\}$$

In particular we have $\sup \emptyset = 0$.

LIMIT ORDINALS

The class of limit ordinals (denoted by *Lim*) is defined by

$$\alpha \in \mathrm{Lim} \iff \alpha \neq 0 \wedge \forall \xi < \alpha : \xi' < \alpha$$

The least limit ordinal is $\omega = \min \mathrm{Lim}$.
For a set $A \subseteq \mathrm{On}$ define

$$\sup A = \min\{\xi \in \mathrm{On} \mid \forall \alpha \in A : \alpha \leq \xi\}$$

In particular we have $\sup \emptyset = 0$.

SUPREMA

Lemma

Assume that $A \neq \emptyset$ and that $A \subseteq \mathrm{On}$ is a set. If $\sup A \notin A$ then $\sup A \in \mathrm{Lim}$.

Suprema

Lemma

Assume that $A \neq \emptyset$ and that $A \subseteq \mathrm{On}$ is a set. If $\sup A \notin A$ then $\sup A \in \mathrm{Lim}$.

Proof. Let $\alpha = \sup A$, then $\alpha > 0$ since $A \neq \emptyset$ and $\sup A \notin A$.

SUPREMA

Lemma

Assume that $A \neq \emptyset$ and that $A \subseteq \mathrm{On}$ is a set. If $\sup A \notin A$ then $\sup A \in \mathrm{Lim}$.

Proof. Let $\alpha = \sup A$, then $\alpha > 0$ since $A \neq \emptyset$ and $\sup A \notin A$. Let $\beta < \alpha$ in A.

SUPREMA

Lemma

Assume that $A \neq \emptyset$ and that $A \subseteq \mathrm{On}$ is a set. If $\sup A \notin A$ then $\sup A \in \mathrm{Lim}$.

Proof. Let $\alpha = \sup A$, then $\alpha > 0$ since $A \neq \emptyset$ and $\sup A \notin A$. Let $\beta < \alpha$ in A. We have to show that $\beta' < \alpha$.

SUPREMA

Lemma

Assume that $A \neq \emptyset$ and that $A \subseteq \mathrm{On}$ is a set. If $\sup A \notin A$ then $\sup A \in \mathrm{Lim}$.

Proof. Let $\alpha = \sup A$, then $\alpha > 0$ since $A \neq \emptyset$ and $\sup A \notin A$. Let $\beta < \alpha$ in A. We have to show that $\beta' < \alpha$. Because of $\beta < \alpha$ and $\alpha = \sup A$ there exists a $\xi \in A$ such that $\beta < \xi < \alpha$ and thus $\beta' \leq \xi < \alpha$.

WELL FOUNDED RELATIONS

A binary relation R is called well founded if every non empty set contains an R-minimal element.

WELL FOUNDED RELATIONS

A binary relation R is called well founded if every non empty set contains an R-minimal element.

Lemma

1. If $\prec$ is well founded then there does not exist an infinite descending chain of elements in with respect to $\prec$.

UNIVERSITEIT GENT

Well founded relations

A binary relation R is called well founded if every non empty set contains an R-minimal element.

Lemma

1. If $\prec$ is well founded then there does not exist an infinite descending chain of elements in with respect to $\prec$.
2. If $\prec \subseteq A \times A$ and if there exists an $F : A \to \mathrm{On}$ such that $\forall x, y : x \prec y \to F(x) < F(y)$ then $\prec$ is well founded.

WELL FOUNDED RELATIONS

A binary relation R is called well founded if every non empty set contains an R-minimal element.

Lemma

1. If $\prec$ is well founded then there does not exist an infinite descending chain of elements in with respect to $\prec$.
2. If $\prec \subseteq A \times A$ and if there exists an $F : A \to \mathrm{On}$ such that $\forall x, y : x \prec y \to F(x) < F(y)$ then $\prec$ is well founded.

Proof.
The first assertion is obvious. The elements of an infinite descending chain form a non empty set without a minimal element.

For a proof of the second assertion assume that $X \neq \emptyset$. If $X \cap A = \emptyset$ then every $x \in X$ is a $<$-minimal element and the assertion follows.

For a proof of the second assertion assume that $X \neq \emptyset$. If $X \cap A = \emptyset$ then every $x \in X$ is a $<$-minimal element and the assertion follows.
Assume now that $X \cap A \neq \emptyset$ and define $\beta = \min\{F(x) \mid x \in X \cap A\}$.

For a proof of the second assertion assume that $X \neq \emptyset$. If $X \cap A = \emptyset$ then every $x \in X$ is a $<$-minimal element and the assertion follows.

Assume now that $X \cap A \neq \emptyset$ and define $\beta = \min\{F(x) \mid x \in X \cap A\}$. Let $x_0 \in X \cap A$ such that $F(x_0) = \beta$.

For a proof of the second assertion assume that $X \neq \emptyset$. If $X \cap A = \emptyset$ then every $x \in X$ is a $<$-minimal element and the assertion follows.
Assume now that $X \cap A \neq \emptyset$ and define $\beta = \min\{F(x) \mid x \in X \cap A\}$. Let $x_0 \in X \cap A$ such that $F(x_0) = \beta$. Then x_0 is minimal.

For a proof of the second assertion assume that $X \neq \emptyset$. If $X \cap A = \emptyset$ then every $x \in X$ is a $<$-minimal element and the assertion follows.
Assume now that $X \cap A \neq \emptyset$ and define $\beta = \min\{F(x) \mid x \in X \cap A\}$. Let $x_0 \in X \cap A$ such that $F(x_0) = \beta$. Then x_0 is minimal. If there exists an $x_1 \in X$ such that $x_1 < x_0$ then necessarily $x_1 \notin A$.

For a proof of the second assertion assume that $X \neq \emptyset$. If $X \cap A = \emptyset$ then every $x \in X$ is a $<$-minimal element and the assertion follows.
Assume now that $X \cap A \neq \emptyset$ and define $\beta = \min\{F(x) \mid x \in X \cap A\}$. Let $x_0 \in X \cap A$ such that $F(x_0) = \beta$. Then x_0 is minimal. If there exists an $x_1 \in X$ such that $x_1 < x_0$ then necessarily $x_1 \notin A$. Otherwise $x_1 \in A$ yields $F(x_1) < F(x_0)$ in contradiction with the assumption that x_0 is minimal.

A function $F : \mathrm{On} \to \mathrm{On}$ is called order preserving (o.p.) if $\alpha < \beta \implies F(\alpha) < F(\beta)$.

A function $F : \mathrm{On} \to \mathrm{On}$ is called order preserving (o.p.) if $\alpha < \beta \implies F(\alpha) < F(\beta)$.

Lemma

If $F : \mathrm{On} \to \mathrm{On}$ is o.p., then $F(\alpha) \geq \alpha$ holds for all $\alpha \in \mathrm{On}$.

A function $F : \mathrm{On} \to \mathrm{On}$ is called order preserving (o.p.) if $\alpha < \beta \implies F(\alpha) < F(\beta)$.

Lemma

If $F : \mathrm{On} \to \mathrm{On}$ is o.p., then $F(\alpha) \geq \alpha$ holds for all $\alpha \in \mathrm{On}$.

Proof. Assume that there exists an $\alpha \in \mathrm{On}$ such that $F(\alpha) < \alpha$. Define

$$C = \{\xi \in \mathrm{On} \mid F(\xi) < \xi\}.$$

Then $C \neq \emptyset$ and $C \subseteq \mathrm{On}$, hence there exists $\alpha_0 = \min C$. Since F is order preserving we conclude from $F(\alpha_0) < \alpha_0$ that $F(F(\alpha_0)) < F(\alpha_0)$ and thus $F(\alpha_0) \in C$. But we have $F(\alpha_0) < \alpha_0$ in contradiction with the minimality of α_0.

A function $F : \mathrm{On} \to \mathrm{On}$ is called order preserving (o.p.) if $\alpha < \beta \implies F(\alpha) < F(\beta)$.

Lemma

If $F : \mathrm{On} \to \mathrm{On}$ is o.p., then $F(\alpha) \geq \alpha$ holds for all $\alpha \in \mathrm{On}$.

Proof. Assume that there exists an $\alpha \in \mathrm{On}$ such that $F(\alpha) < \alpha$. Define

$$C = \{\xi \in \mathrm{On} \mid F(\xi) < \xi\}.$$

Then $C \neq \emptyset$ and $C \subseteq \mathrm{On}$, hence there exists $\alpha_0 = \min C$.

A function $F : \mathrm{On} \to \mathrm{On}$ is called order preserving (o.p.) if $\alpha < \beta \implies F(\alpha) < F(\beta)$.

Lemma

If $F : \mathrm{On} \to \mathrm{On}$ is o.p., then $F(\alpha) \geq \alpha$ holds for all $\alpha \in \mathrm{On}$.

Proof. Assume that there exists an $\alpha \in \mathrm{On}$ such that $F(\alpha) < \alpha$. Define

$$C = \{\xi \in \mathrm{On} \mid F(\xi) < \xi\}.$$

Then $C \neq \emptyset$ and $C \subseteq \mathrm{On}$, hence there exists $\alpha_0 = \min C$. Since F is order preserving we conclude from $F(\alpha_0) < \alpha_0$ that $F(F(\alpha_0)) < F(\alpha_0)$

A function $F : \mathrm{On} \to \mathrm{On}$ is called order preserving (o.p.) if $\alpha < \beta \implies F(\alpha) < F(\beta)$.

Lemma

If $F : \mathrm{On} \to \mathrm{On}$ is o.p., then $F(\alpha) \geq \alpha$ holds for all $\alpha \in \mathrm{On}$.

Proof. Assume that there exists an $\alpha \in \mathrm{On}$ such that $F(\alpha) < \alpha$. Define

$$C = \{\xi \in \mathrm{On} \mid F(\xi) < \xi\}.$$

Then $C \neq \emptyset$ and $C \subseteq \mathrm{On}$, hence there exists $\alpha_0 = \min C$. Since F is order preserving we conclude from $F(\alpha_0) < \alpha_0$ that $F(F(\alpha_0)) < F(\alpha_0)$ and thus $F(\alpha_0) \in C$. But we have $F(\alpha_0) < \alpha_0$ in contradiction with the minimality of α_0.

ENUMERATION FUNCTIONS

A function $F : \mathrm{On} \to A$ is called ordering function for $A \subseteq \mathrm{On}$ if F is order preserving and surjective onto A.

ENUMERATION FUNCTIONS

A function $F : \mathrm{On} \to A$ is called ordering function for $A \subseteq \mathrm{On}$ if F is order preserving and surjective onto A.
$A \subseteq \mathrm{On}$ is unbounded if for all $\alpha \in \mathrm{On}$ there exists a $\beta \in A$ such that $\alpha < \beta$.

Enumeration functions

A function $F : \mathrm{On} \to A$ is called ordering function for $A \subseteq \mathrm{On}$ if F is order preserving and surjective onto A.
$A \subseteq \mathrm{On}$ is unbounded if for all $\alpha \in \mathrm{On}$ there exists a $\beta \in A$ such that $\alpha < \beta$.
$A \subseteq \mathrm{On}$ is called closed if $\sup X \in A$ for all non empty sets $X \subseteq A$.

ENUMERATION FUNCTIONS

A function $F : \mathrm{On} \to A$ is called ordering function for $A \subseteq \mathrm{On}$ if F is order preserving and surjective onto A.
$A \subseteq \mathrm{On}$ is unbounded if for all $\alpha \in \mathrm{On}$ there exists a $\beta \in A$ such that $\alpha < \beta$.
$A \subseteq \mathrm{On}$ is called closed if $\sup X \in A$ for all non empty sets $X \subseteq A$.
If $A \subseteq \mathrm{On}$ is closed and unbounded then A is called club.

ENUMERATION FUNCTIONS

Lemma

Let A be unbounded. Then there exists a uniquely determined ordering function Enum_A on A such that

$$\mathrm{Enum}_A(\alpha) = \min\{\beta \in A \mid \forall \xi < \alpha : \mathrm{Enum}_A(\xi) < \beta\}.$$

Enumeration functions

Lemma

Let A be unbounded. Then there exists a uniquely determined ordering function Enum_A on A such that

$$\mathrm{Enum}_A(\alpha) = \min\{\beta \in A \mid \forall \xi < \alpha : \mathrm{Enum}_A(\xi) < \beta\}.$$

Proof of existence of F for A using transfinite recursion.

Enumeration functions

Lemma

Let A be unbounded. Then there exists a uniquely determined ordering function Enum_A on A such that

$$\mathrm{Enum}_A(\alpha) = \min\{\beta \in A \mid \forall \xi < \alpha : \mathrm{Enum}_A(\xi) < \beta\}.$$

Proof of existence of F for A using transfinite recursion. Assume inductively that $F(\xi)$ is defined for all $\xi < \alpha$.

ENUMERATION FUNCTIONS

Lemma

Let A be unbounded. Then there exists a uniquely determined ordering function Enum_A on A such that

$$\mathrm{Enum}_A(\alpha) = \min\{\beta \in A \mid \forall \xi < \alpha : \mathrm{Enum}_A(\xi) < \beta\}.$$

Proof of existence of F for A using transfinite recursion. Assume inductively that $F(\xi)$ is defined for all $\xi < \alpha$. Then $\{F(\xi) \mid \xi < \alpha\}$ is a set and hence there exists a minimal $\gamma \in \mathrm{On}$ such that $F(\xi) < \gamma$ for all $\xi < \alpha$.

ENUMERATION FUNCTIONS

Lemma

Let A be unbounded. Then there exists a uniquely determined ordering function Enum_A on A such that

$$\mathrm{Enum}_A(\alpha) = \min\{\beta \in A \mid \forall \xi < \alpha : \mathrm{Enum}_A(\xi) < \beta\}.$$

Proof of existence of F for A using transfinite recursion. Assume inductively that $F(\xi)$ is defined for all $\xi < \alpha$. Then $\{F(\xi) \mid \xi < \alpha\}$ is a set and hence there exists a minimal $\gamma \in \mathrm{On}$ such that $F(\xi) < \gamma$ for all $\xi < \alpha$. Because A is unbounded there exists a minimal $\beta \in A$ such that $\gamma \leq \beta$.

Enumeration functions

Lemma

Let A be unbounded. Then there exists a uniquely determined ordering function Enum_A on A such that

$$\mathrm{Enum}_A(\alpha) = \min\{\beta \in A \mid \forall \xi < \alpha : \mathrm{Enum}_A(\xi) < \beta\}.$$

Proof of existence of F for A using transfinite recursion. Assume inductively that $F(\xi)$ is defined for all $\xi < \alpha$. Then $\{F(\xi) \mid \xi < \alpha\}$ is a set and hence there exists a minimal $\gamma \in \mathrm{On}$ such that $F(\xi) < \gamma$ for all $\xi < \alpha$. Because A is unbounded there exists a minimal $\beta \in A$ such that $\gamma \leq \beta$. Hence $F(\alpha) = \beta$ and so F is totally defined.

We still have to show that F is order preserving and surjective with $\mathrm{range}(F) = A$.

We still have to show that F is order preserving and surjective with $\mathrm{range}(F) = A$. That F is order preserving is easy to see. To prove surjectivity let $\gamma \in A$.

We still have to show that F is order preserving and surjective with $\mathrm{range}(F) = A$. That F is order preserving is easy to see. To prove surjectivity let $\gamma \in A$. Then there exists $\alpha = \min\{\xi \mid \gamma \leq F(\xi)\}$ since $\alpha \leq F(\alpha)$.

We still have to show that F is order preserving and surjective with $\mathrm{range}(F) = A$. That F is order preserving is easy to see. To prove surjectivity let $\gamma \in A$. Then there exists $\alpha = \min\{\xi \mid \gamma \leq F(\xi)\}$ since $\alpha \leq F(\alpha)$. We have $\gamma \leq F(\alpha)$ and $\forall \xi < \alpha : F(\xi) < \gamma$ so that $F(\alpha) = \gamma$.

Proof of uniqueness. Let F and G both be ordering functions for A, thus $F, G : \mathrm{On} \to A$ and F, G are surjective and order preserving.

Proof of uniqueness. Let F and G both be ordering functions for A, thus $F, G : \mathrm{On} \to A$ and F, G are surjective and order preserving. Assume by induction on α that $\forall \xi < \alpha : F(\xi) = G(\xi)$.

Proof of uniqueness. Let F and G both be ordering functions for A, thus $F, G : \mathrm{On} \to A$ and F, G are surjective and order preserving. Assume by induction on α that $\forall \xi < \alpha : F(\xi) = G(\xi)$. Assume for a contradiction that $F(\alpha) \neq G(\alpha)$. We have then two cases:

Proof of uniqueness. Let F and G both be ordering functions for A, thus $F, G : \mathrm{On} \to A$ and F, G are surjective and order preserving. Assume by induction on α that $\forall \xi < \alpha : F(\xi) = G(\xi)$. Assume for a contradiction that $F(\alpha) \neq G(\alpha)$. We have then two cases:

- $F(\alpha) < G(\alpha)$.

Proof of uniqueness. Let F and G both be ordering functions for A, thus $F, G : \mathrm{On} \to A$ and F, G are surjective and order preserving. Assume by induction on α that $\forall \xi < \alpha : F(\xi) = G(\xi)$. Assume for a contradiction that $F(\alpha) \neq G(\alpha)$. We have then two cases:

- $F(\alpha) < G(\alpha)$. The surjectivity of G yields that there exists a β such that $F(\alpha) = G(\beta)$.

Proof of uniqueness. Let F and G both be ordering functions for A, thus $F, G : \mathrm{On} \to A$ and F, G are surjective and order preserving. Assume by induction on α that $\forall \xi < \alpha : F(\xi) = G(\xi)$. Assume for a contradiction that $F(\alpha) \neq G(\alpha)$. We have then two cases:

- $F(\alpha) < G(\alpha)$. The surjectivity of G yields that there exists a β such that $F(\alpha) = G(\beta)$. Then $\beta > \alpha$ since $\beta = \alpha$ is impossible by assumption and if $\beta < \alpha$ the induction hypothesis yields that $F(\beta) = G(\beta) = F(\alpha)$.

Proof of uniqueness. Let F and G both be ordering functions for A, thus $F, G : \mathrm{On} \to A$ and F, G are surjective and order preserving. Assume by induction on α that $\forall \xi < \alpha : F(\xi) = G(\xi)$. Assume for a contradiction that $F(\alpha) \neq G(\alpha)$. We have then two cases:

- $F(\alpha) < G(\alpha)$. The surjectivity of G yields that there exists a β such that $F(\alpha) = G(\beta)$. Then $\beta > \alpha$ since $\beta = \alpha$ is impossible by assumption and if $\beta < \alpha$ the induction hypothesis yields that $F(\beta) = G(\beta) = F(\alpha)$. But F is order preserving hence $\beta < \alpha$ yields $F(\beta) < F(\alpha)$, contradiction. Thus $\beta > \alpha$ and the order preservation of G yields $G(\beta) > G(\alpha) > F(\alpha)$.

Proof of uniqueness. Let F and G both be ordering functions for A, thus $F, G : \mathrm{On} \to A$ and F, G are surjective and order preserving. Assume by induction on α that $\forall \xi < \alpha : F(\xi) = G(\xi)$. Assume for a contradiction that $F(\alpha) \neq G(\alpha)$. We have then two cases:

- $F(\alpha) < G(\alpha)$. The surjectivity of G yields that there exists a β such that $F(\alpha) = G(\beta)$. Then $\beta > \alpha$ since $\beta = \alpha$ is impossible by assumption and if $\beta < \alpha$ the induction hypothesis yields that $F(\beta) = G(\beta) = F(\alpha)$. But F is order preserving hence $\beta < \alpha$ yields $F(\beta) < F(\alpha)$, contradiction. Thus $\beta > \alpha$ and the order preservation of G yields $G(\beta) > G(\alpha) > F(\alpha)$.
- $F(\alpha) > G(\alpha)$. Similarly.

Proof of uniqueness. Let F and G both be ordering functions for A, thus $F, G : \mathrm{On} \to A$ and F, G are surjective and order preserving. Assume by induction on α that $\forall \xi < \alpha : F(\xi) = G(\xi)$. Assume for a contradiction that $F(\alpha) \neq G(\alpha)$. We have then two cases:

- $F(\alpha) < G(\alpha)$. The surjectivity of G yields that there exists a β such that $F(\alpha) = G(\beta)$. Then $\beta > \alpha$ since $\beta = \alpha$ is impossible by assumption and if $\beta < \alpha$ the induction hypothesis yields that $F(\beta) = G(\beta) = F(\alpha)$. But F is order preserving hence $\beta < \alpha$ yields $F(\beta) < F(\alpha)$, contradiction. Thus $\beta > \alpha$ and the order preservation of G yields $G(\beta) > G(\alpha) > F(\alpha)$.
- $F(\alpha) > G(\alpha)$. Similarly.

We conclude $\forall \alpha \in \mathrm{On} : F(\alpha) = G(\alpha)$ hence $F = G$.

Normal functions

A function F is called continuous if
$\forall\lambda \in \mathrm{Lim} : F(\lambda) = \sup\{F(\xi) \mid \xi < \lambda\}$.

Normal functions

A function F is called continuous if
$\forall\lambda \in \mathrm{Lim} : F(\lambda) = \sup\{F(\xi) \mid \xi < \lambda\}$.
A function F is called normal if F is continues and order preserving.

Normal functions

A function F is called continuous if
$\forall\lambda \in \mathrm{Lim} : F(\lambda) = \sup\{F(\xi) \mid \xi < \lambda\}$.
A function F is called normal if F is continues and order preserving.

Lemma

If $F : \mathrm{On} \to \mathrm{On}$ is continuous and if $\forall\alpha : F(\alpha) < F(\alpha')$. Then F is normal.

Normal functions

A function F is called continuous if
$\forall\lambda \in \mathrm{Lim} : F(\lambda) = \sup\{F(\xi) \mid \xi < \lambda\}$.
A function F is called normal if F is continues and order preserving.

Lemma

If $F : \mathrm{On} \to \mathrm{On}$ is continuous and if $\forall\alpha : F(\alpha) < F(\alpha')$. Then F is normal.

Proof. By induction on α we show that F is order preserving, hence $\forall\beta \in \mathrm{On} : \beta < \alpha \implies F(\beta) < F(\alpha)$. The case $\alpha = 0$ is trivial.

Normal functions

A function F is called continuous if
$\forall \lambda \in \mathrm{Lim} : F(\lambda) = \sup\{F(\xi) \mid \xi < \lambda\}$.
A function F is called normal if F is continues and order preserving.

Lemma

If $F : \mathrm{On} \to \mathrm{On}$ is continuous and if $\forall \alpha : F(\alpha) < F(\alpha')$. Then F is normal.

Proof. By induction on α we show that F is order preserving, hence $\forall \beta \in \mathrm{On} : \beta < \alpha \implies F(\beta) < F(\alpha)$. The case $\alpha = 0$ is trivial. Assume that $\alpha = \gamma'$. Then the two cases follow from $\beta < \alpha = \gamma'$:

- $\beta = \gamma$. Then $F(\beta) < F(\alpha)$ follows from the assumption that $F(\beta) < F(\beta')$.

- $\beta = \gamma$. Then $F(\beta) < F(\alpha)$ follows from the assumption that $F(\beta) < F(\beta')$.
- $\beta < \gamma$. Then the induction hypothesis yields that $F(\beta) < F(\gamma)$ and hence $F(\beta) < F(\alpha)$ since $F(\gamma) < F(\gamma')$ is valid by assumption.

- $\beta = \gamma$. Then $F(\beta) < F(\alpha)$ follows from the assumption that $F(\beta) < F(\beta')$.
- $\beta < \gamma$. Then the induction hypothesis yields that $F(\beta) < F(\gamma)$ and hence $F(\beta) < F(\alpha)$ since $F(\gamma) < F(\gamma')$ is valid by assumption.

Assume now that $\alpha \in \mathrm{Lim}$. Since α is a limit we obtain from $\beta < \alpha$ also $\beta' < \alpha$.

- $\beta = \gamma$. Then $F(\beta) < F(\alpha)$ follows from the assumption that $F(\beta) < F(\beta')$.
- $\beta < \gamma$. Then the induction hypothesis yields that $F(\beta) < F(\gamma)$ and hence $F(\beta) < F(\alpha)$ since $F(\gamma) < F(\gamma')$ is valid by assumption.

Assume now that $\alpha \in \mathrm{Lim}$. Since α is a limit we obtain from $\beta < \alpha$ also $\beta' < \alpha$. The continuity of F implies $F(\alpha) = \sup\{F(\xi) \mid \xi < \alpha\}$.

- $\beta = \gamma$. Then $F(\beta) < F(\alpha)$ follows from the assumption that $F(\beta) < F(\beta')$.
- $\beta < \gamma$. Then the induction hypothesis yields that $F(\beta) < F(\gamma)$ and hence $F(\beta) < F(\alpha)$ since $F(\gamma) < F(\gamma')$ is valid by assumption.

Assume now that $\alpha \in \mathrm{Lim}$. Since α is a limit we obtain from $\beta < \alpha$ also $\beta' < \alpha$. The continuity of F implies $F(\alpha) = \sup\{F(\xi) \mid \xi < \alpha\}$. The induction hypothesis yields $F(\beta') \leq F(\alpha)$.

- $\beta = \gamma$. Then $F(\beta) < F(\alpha)$ follows from the assumption that $F(\beta) < F(\beta')$.
- $\beta < \gamma$. Then the induction hypothesis yields that $F(\beta) < F(\gamma)$ and hence $F(\beta) < F(\alpha)$ since $F(\gamma) < F(\gamma')$ is valid by assumption.

Assume now that $\alpha \in \mathrm{Lim}$. Since α is a limit we obtain from $\beta < \alpha$ also $\beta' < \alpha$. The continuity of F implies $F(\alpha) = \sup\{F(\xi) \mid \xi < \alpha\}$. The induction hypothesis yields $F(\beta') \leq F(\alpha)$. By assumption we have $F(\beta) < F(\beta')$ hence $F(\beta) < F(\alpha)$.

Elementary properties of normal functions

Lemma

Let F be normal.

Elementary properties of normal functions

Lemma

Let F be normal.

1. $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$ if $\alpha > 0$.

Elementary properties of normal functions

Lemma

Let F be normal.

1. $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$ if $\alpha > 0$.
2. If $\lambda \in \mathrm{Lim}$ then $F(\lambda) \in \mathrm{Lim}$.

Elementary properties of normal functions

Lemma

Let F be normal.

1. $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$ if $\alpha > 0$.
2. If $\lambda \in \mathrm{Lim}$ then $F(\lambda) \in \mathrm{Lim}$.
3. For $\gamma \geq F(0)$ there exists a uniquely determined α such that $F(\alpha) \leq \gamma < F(\alpha')$.

Elementary properties of normal functions

Lemma

Let F be normal.

1. $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$ if $\alpha > 0$.
2. If $\lambda \in \mathrm{Lim}$ then $F(\lambda) \in \mathrm{Lim}$.
3. For $\gamma \geq F(0)$ there exists a uniquely determined α such that $F(\alpha) \leq \gamma < F(\alpha')$.
4. Let G be normal. Then $F \circ G$ is normal, too.

Elementary properties of normal functions

Lemma

Let F be normal.

1. $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$ if $\alpha > 0$.
2. If $\lambda \in \mathrm{Lim}$ then $F(\lambda) \in \mathrm{Lim}$.
3. For $\gamma \geq F(0)$ there exists a uniquely determined α such that $F(\alpha) \leq \gamma < F(\alpha')$.
4. Let G be normal. Then $F \circ G$ is normal, too.
5. For a non empty set A we have $F(\sup A) = \sup F[A]$ where $F[A] = \{F(\alpha) \mid \alpha \in A\}$.

Proof of the first assertion. Assume inductively that $F(\beta) = \sup\{F(\xi') \mid \xi < \beta\}$ for all $\beta < \alpha$.

Proof of the first assertion. Assume inductively that $F(\beta) = \sup\{F(\xi') \mid \xi < \beta\}$ for all $\beta < \alpha$. Assume first that α is a successor.

Proof of the first assertion. Assume inductively that $F(\beta) = \sup\{F(\xi') \mid \xi < \beta\}$ for all $\beta < \alpha$. Assume first that α is a successor. We have $F(\alpha) \in \{F(\xi') \mid \xi < \alpha\}$ thence $F(\alpha) \leq \sup\{F(\xi') \mid \xi < \alpha\}$. F is order preserving, hence $\xi' \leq \alpha \implies F(\xi') \leq F(\alpha)$ so that $\sup\{F(\xi') \mid \xi < \alpha\} \leq F(\alpha)$.

Proof of the first assertion. Assume inductively that $F(\beta) = \sup\{F(\xi') \mid \xi < \beta\}$ for all $\beta < \alpha$. Assume first that α is a successor. We have $F(\alpha) \in \{F(\xi') \mid \xi < \alpha\}$ thence $F(\alpha) \leq \sup\{F(\xi') \mid \xi < \alpha\}$. F is order preserving, hence $\xi' \leq \alpha \implies F(\xi') \leq F(\alpha)$ so that $\sup\{F(\xi') \mid \xi < \alpha\} \leq F(\alpha)$. Therefore $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$.

Proof of the first assertion. Assume inductively that $F(\beta) = \sup\{F(\xi') \mid \xi < \beta\}$ for all $\beta < \alpha$. Assume first that α is a successor. We have $F(\alpha) \in \{F(\xi') \mid \xi < \alpha\}$ thence $F(\alpha) \leq \sup\{F(\xi') \mid \xi < \alpha\}$. F is order preserving, hence $\xi' \leq \alpha \implies F(\xi') \leq F(\alpha)$ so that $\sup\{F(\xi') \mid \xi < \alpha\} \leq F(\alpha)$. Therefore $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$.
Let $\alpha \in \mathrm{Lim}$. Then
$F(\alpha) = \sup\{F(\xi) \mid \xi < \alpha\} = \sup\{F(\xi') \mid \xi < \alpha\}$.

Proof of the first assertion. Assume inductively that $F(\beta) = \sup\{F(\xi') \mid \xi < \beta\}$ for all $\beta < \alpha$. Assume first that α is a successor. We have $F(\alpha) \in \{F(\xi') \mid \xi < \alpha\}$ thence $F(\alpha) \leq \sup\{F(\xi') \mid \xi < \alpha\}$. F is order preserving, hence $\xi' \leq \alpha \implies F(\xi') \leq F(\alpha)$ so that $\sup\{F(\xi') \mid \xi < \alpha\} \leq F(\alpha)$.
Therefore $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$.
Let $\alpha \in \mathrm{Lim}$. Then
$F(\alpha) = \sup\{F(\xi) \mid \xi < \alpha\} = \sup\{F(\xi') \mid \xi < \alpha\}$.
Proof of the second assertion. Let $\lambda \in \mathrm{Lim}$. Then $F(\lambda) = \sup\{F(\xi) \mid \xi < \lambda\}$ because F is continuous.

Proof of the first assertion. Assume inductively that $F(\beta) = \sup\{F(\xi') \mid \xi < \beta\}$ for all $\beta < \alpha$. Assume first that α is a successor. We have $F(\alpha) \in \{F(\xi') \mid \xi < \alpha\}$ thence $F(\alpha) \leq \sup\{F(\xi') \mid \xi < \alpha\}$. F is order preserving, hence $\xi' \leq \alpha \implies F(\xi') \leq F(\alpha)$ so that $\sup\{F(\xi') \mid \xi < \alpha\} \leq F(\alpha)$.
Therefore $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$.
Let $\alpha \in \mathrm{Lim}$. Then
$F(\alpha) = \sup\{F(\xi) \mid \xi < \alpha\} = \sup\{F(\xi') \mid \xi < \alpha\}$.
Proof of the second assertion. Let $\lambda \in \mathrm{Lim}$. Then $F(\lambda) = \sup\{F(\xi) \mid \xi < \lambda\}$ because F is continuous. Suppose $\gamma < F(\lambda)$.

Proof of the first assertion. Assume inductively that $F(\beta) = \sup\{F(\xi') \mid \xi < \beta\}$ for all $\beta < \alpha$. Assume first that α is a successor. We have $F(\alpha) \in \{F(\xi') \mid \xi < \alpha\}$ thence $F(\alpha) \leq \sup\{F(\xi') \mid \xi < \alpha\}$. F is order preserving, hence $\xi' \leq \alpha \implies F(\xi') \leq F(\alpha)$ so that $\sup\{F(\xi') \mid \xi < \alpha\} \leq F(\alpha)$. Therefore $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$.
Let $\alpha \in \mathrm{Lim}$. Then
$F(\alpha) = \sup\{F(\xi) \mid \xi < \alpha\} = \sup\{F(\xi') \mid \xi < \alpha\}$.
Proof of the second assertion. Let $\lambda \in \mathrm{Lim}$. Then $F(\lambda) = \sup\{F(\xi) \mid \xi < \lambda\}$ because F is continuous. Suppose $\gamma < F(\lambda)$. Then $\gamma < F(\xi)$ for some $\xi < \lambda$. Then $\gamma' \leq F(\xi) < F(\xi') < F(\lambda)$ because F is order preserving.

Proof of the first assertion. Assume inductively that $F(\beta) = \sup\{F(\xi') \mid \xi < \beta\}$ for all $\beta < \alpha$. Assume first that α is a successor. We have $F(\alpha) \in \{F(\xi') \mid \xi < \alpha\}$ thence $F(\alpha) \leq \sup\{F(\xi') \mid \xi < \alpha\}$. F is order preserving, hence $\xi' \leq \alpha \implies F(\xi') \leq F(\alpha)$ so that $\sup\{F(\xi') \mid \xi < \alpha\} \leq F(\alpha)$. Therefore $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$.
Let $\alpha \in \mathrm{Lim}$. Then
$F(\alpha) = \sup\{F(\xi) \mid \xi < \alpha\} = \sup\{F(\xi') \mid \xi < \alpha\}$.
Proof of the second assertion. Let $\lambda \in \mathrm{Lim}$. Then $F(\lambda) = \sup\{F(\xi) \mid \xi < \lambda\}$ because F is continuous. Suppose $\gamma < F(\lambda)$. Then $\gamma < F(\xi)$ for some $\xi < \lambda$. Then $\gamma' \leq F(\xi) < F(\xi') < F(\lambda)$ because F is order preserving.

Proof of assertion three. Let $\gamma \geq F(0)$.

Proof of assertion three. Let $\gamma \geq F(0)$. Then $\gamma \leq F(\gamma) < F(\gamma')$.

Proof of assertion three. Let $\gamma \geq F(0)$. Then $\gamma \leq F(\gamma) < F(\gamma')$. So there exists $\alpha = \min\{\xi \mid \gamma < F(\xi')\}$.

Proof of assertion three. Let $\gamma \geq F(0)$. Then $\gamma \leq F(\gamma) < F(\gamma')$. So there exists $\alpha = \min\{\xi \mid \gamma < F(\xi')\}$. Then $\gamma < F(\alpha')$.

Proof of assertion three. Let $\gamma \geq F(0)$. Then $\gamma \leq F(\gamma) < F(\gamma')$. So there exists $\alpha = \min\{\xi \mid \gamma < F(\xi')\}$. Then $\gamma < F(\alpha')$. If $\alpha = 0$ then $\gamma = F(0)$ en the assertion follows.

UNIVERSITEIT GENT

Proof of assertion three. Let $\gamma \geq F(0)$. Then $\gamma \leq F(\gamma) < F(\gamma')$. So there exists $\alpha = \min\{\xi \mid \gamma < F(\xi')\}$. Then $\gamma < F(\alpha')$. If $\alpha = 0$ then $\gamma = F(0)$ en the assertion follows. If $\alpha > 0$ then $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$.

Proof of assertion three. Let $\gamma \geq F(0)$. Then $\gamma \leq F(\gamma) < F(\gamma')$. So there exists $\alpha = \min\{\xi \mid \gamma < F(\xi')\}$. Then $\gamma < F(\alpha')$. If $\alpha = 0$ then $\gamma = F(0)$ en the assertion follows. If $\alpha > 0$ then $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$. If $\xi < \alpha$ then $F(\xi') \leq \gamma$ so that $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\} \leq \gamma$.

Proof of assertion three. Let $\gamma \geq F(0)$. Then $\gamma \leq F(\gamma) < F(\gamma')$. So there exists $\alpha = \min\{\xi \mid \gamma < F(\xi')\}$. Then $\gamma < F(\alpha')$. If $\alpha = 0$ then $\gamma = F(0)$ en the assertion follows. If $\alpha > 0$ then $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$. If $\xi < \alpha$ then $F(\xi') \leq \gamma$ so that $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\} \leq \gamma$.
Proof of assertion four. This Is easy.

Proof of assertion three. Let $\gamma \geq F(0)$. Then $\gamma \leq F(\gamma) < F(\gamma')$. So there exists $\alpha = \min\{\xi \mid \gamma < F(\xi')\}$. Then $\gamma < F(\alpha')$. If $\alpha = 0$ then $\gamma = F(0)$ en the assertion follows. If $\alpha > 0$ then $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$. If $\xi < \alpha$ then $F(\xi') \leq \gamma$ so that $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\} \leq \gamma$.
Proof of assertion four. This Is easy.
Proof of assertion five. Suppose that $A \subseteq \mathrm{On}$ is a non empty set with $\alpha = \sup A$.

Proof of assertion three. Let $\gamma \geq F(0)$. Then $\gamma \leq F(\gamma) < F(\gamma')$. So there exists $\alpha = \min\{\xi \mid \gamma < F(\xi')\}$. Then $\gamma < F(\alpha')$. If $\alpha = 0$ then $\gamma = F(0)$ en the assertion follows. If $\alpha > 0$ then $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$. If $\xi < \alpha$ then $F(\xi') \leq \gamma$ so that $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\} \leq \gamma$.
Proof of assertion four. This Is easy.
Proof of assertion five. Suppose that $A \subseteq \mathrm{On}$ is a non empty set with $\alpha = \sup A$. If $\alpha \in A$ then $F(\alpha) = F(\sup A) = \sup F[A]$ since F is order preserving.

Proof of assertion three. Let $\gamma \geq F(0)$. Then $\gamma \leq F(\gamma) < F(\gamma')$. So there exists $\alpha = \min\{\xi \mid \gamma < F(\xi')\}$. Then $\gamma < F(\alpha')$. If $\alpha = 0$ then $\gamma = F(0)$ en the assertion follows. If $\alpha > 0$ then $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\}$. If $\xi < \alpha$ then $F(\xi') \leq \gamma$ so that $F(\alpha) = \sup\{F(\xi') \mid \xi < \alpha\} \leq \gamma$.

Proof of assertion four. This Is easy.

Proof of assertion five. Suppose that $A \subseteq \mathrm{On}$ is a non empty set with $\alpha = \sup A$. If $\alpha \in A$ then $F(\alpha) = F(\sup A) = \sup F[A]$ since F is order preserving. If $\alpha \notin A$ then $\alpha \in \mathrm{Lim}$ such that $F(\alpha) = \sup\{F(\xi) \mid \xi < \alpha\} = \sup F[A]$.

Lemma

Let $A \subseteq \mathrm{On}$. Then A is club iff Enum_A is normal.

Fixed point lemma for normal functions

Lemma

If F is normal, then there exists a least α such that $F(\alpha) = \alpha$.

Fixed point lemma for normal functions

Lemma

If F is normal, then there exists a least α such that $F(\alpha) = \alpha$.

Proof. Let us define a sequence α_n as follows.

$$\alpha_0 = 0 \qquad \alpha_{n+1} = F(\alpha_n)$$

Fixed point lemma for normal functions

Lemma

If F is normal, then there exists a least α such that $F(\alpha) = \alpha$.

Proof. Let us define a sequence α_n as follows.

$$\alpha_0 = 0 \qquad\qquad \alpha_{n+1} = F(\alpha_n)$$

Let $\beta := \sup\{\alpha_n \mid n < \omega\}$.

Fixed point lemma for normal functions

Lemma

If F is normal, then there exists a least α such that $F(\alpha) = \alpha$.

Proof. Let us define a sequence α_n as follows.

$$\alpha_0 = 0 \qquad\qquad \alpha_{n+1} = F(\alpha_n)$$

Let $\beta := \sup\{\alpha_n \mid n < \omega\}$. Then

$$\begin{aligned} F(\beta) &= F(\sup\{\alpha_n \mid n < \omega\}) \\ &= \sup\{F(\alpha_n) \mid n < \omega\} \\ &= \sup\{\alpha_{n+1} \mid n < \omega\} \\ &= \beta \end{aligned}$$

hence β is a fixed point of F and there will be a smallest one.

Lemma

If F is normal, then $\{\alpha \in On : F(\alpha) = \alpha\}$ is club.

The sum of ordinals

The sum of ordinals is defined by transfinite recursion:

$$\alpha + 0 = \alpha$$
$$\alpha + \beta' = (\alpha + \beta)'$$
$$\alpha + \lambda = \sup\{\alpha + \xi \mid \xi < \lambda\}$$

UNIVERSITEIT GENT

Lemma

1. The function $\beta \mapsto \alpha + \beta$ is normal.

Lemma

1. The function $\beta \mapsto \alpha + \beta$ is normal.
2. $\beta_0 < \beta_1 \implies \alpha + \beta_0 < \alpha + \beta_1$.

Lemma

1. The function $\beta \mapsto \alpha + \beta$ is normal.
2. $\beta_0 < \beta_1 \implies \alpha + \beta_0 < \alpha + \beta_1$.
3. $\alpha, \beta \leq \alpha + \beta$.

Lemma

1. The function $\beta \mapsto \alpha + \beta$ is normal.
2. $\beta_0 < \beta_1 \implies \alpha + \beta_0 < \alpha + \beta_1$.
3. $\alpha, \beta \leq \alpha + \beta$.
4. For all $\gamma \geq \alpha$ there exists a unique β such that $\gamma = \alpha + \beta$.

Lemma

1. The function $\beta \mapsto \alpha + \beta$ is normal.
2. $\beta_0 < \beta_1 \implies \alpha + \beta_0 < \alpha + \beta_1$.
3. $\alpha, \beta \leq \alpha + \beta$.
4. For all $\gamma \geq \alpha$ there exists a unique β such that $\gamma = \alpha + \beta$.
5. $\alpha_0 \leq \alpha_1 \implies \alpha_0 + \beta \leq \alpha_1 + \beta$.

Lemma

1. The function $\beta \mapsto \alpha + \beta$ is normal.
2. $\beta_0 < \beta_1 \implies \alpha + \beta_0 < \alpha + \beta_1$.
3. $\alpha, \beta \leq \alpha + \beta$.
4. For all $\gamma \geq \alpha$ there exists a unique β such that $\gamma = \alpha + \beta$.
5. $\alpha_0 \leq \alpha_1 \implies \alpha_0 + \beta \leq \alpha_1 + \beta$.
6. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

Lemma

1. The function $\beta \mapsto \alpha + \beta$ is normal.
2. $\beta_0 < \beta_1 \implies \alpha + \beta_0 < \alpha + \beta_1$.
3. $\alpha, \beta \leq \alpha + \beta$.
4. For all $\gamma \geq \alpha$ there exists a unique β such that $\gamma = \alpha + \beta$.
5. $\alpha_0 \leq \alpha_1 \implies \alpha_0 + \beta \leq \alpha_1 + \beta$.
6. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
7. $\alpha, \beta < \omega \implies \alpha + \beta = \beta + \alpha$.

Lemma

1. The function $\beta \mapsto \alpha + \beta$ is normal.
2. $\beta_0 < \beta_1 \implies \alpha + \beta_0 < \alpha + \beta_1$.
3. $\alpha, \beta \leq \alpha + \beta$.
4. For all $\gamma \geq \alpha$ there exists a unique β such that $\gamma = \alpha + \beta$.
5. $\alpha_0 \leq \alpha_1 \implies \alpha_0 + \beta \leq \alpha_1 + \beta$.
6. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
7. $\alpha, \beta < \omega \implies \alpha + \beta = \beta + \alpha$.
8. $0 < k < \omega \implies k + \omega = \omega < \omega + k$.

Proof. Define $F(\beta) = \alpha + \beta$ for a given α.

1. F continuous by definition. Note that $\alpha + \beta < (\alpha + \beta)' = \alpha + \beta'$ so that $\forall\beta : F(\beta) < F(\beta')$. This yields that F is normal.

Proof. Define $F(\beta) = \alpha + \beta$ for a given α.

1. F continuous by definition. Note that $\alpha + \beta < (\alpha + \beta)' = \alpha + \beta'$ so that $\forall \beta : F(\beta) < F(\beta')$. This yields that F is normal.
2. This assertion follows from the normality of F.

Proof. Define $F(\beta) = \alpha + \beta$ for a given α.

1. F continuous by definition. Note that $\alpha + \beta < (\alpha + \beta)' = \alpha + \beta'$ so that $\forall \beta : F(\beta) < F(\beta')$. This yields that F is normal.
2. This assertion follows from the normality of F.
3. Since F is normal we see $\beta \leq \alpha + \beta$.

Proof. Define $F(\beta) = \alpha + \beta$ for a given α.

1. F continuous by definition. Note that $\alpha + \beta < (\alpha + \beta)' = \alpha + \beta'$ so that $\forall \beta : F(\beta) < F(\beta')$. This yields that F is normal.
2. This assertion follows from the normality of F.
3. Since F is normal we see $\beta \leq \alpha + \beta$. The other assertion follows by Induction on β.

Proof. Define $F(\beta) = \alpha + \beta$ for a given α.

1. F continuous by definition. Note that $\alpha + \beta < (\alpha + \beta)' = \alpha + \beta'$ so that $\forall \beta : F(\beta) < F(\beta')$. This yields that F is normal.
2. This assertion follows from the normality of F.
3. Since F is normal we see $\beta \leq \alpha + \beta$. The other assertion follows by Induction on β.
4. Suppose $\gamma \geq \alpha$ and choose a β such that $\alpha + \beta \leq \gamma < \alpha + \beta'$. Then $\gamma = \alpha + \beta$.

Proof. Define $F(\beta) = \alpha + \beta$ for a given α.

1. F continuous by definition. Note that $\alpha + \beta < (\alpha + \beta)' = \alpha + \beta'$ so that $\forall \beta : F(\beta) < F(\beta')$. This yields that F is normal.
2. This assertion follows from the normality of F.
3. Since F is normal we see $\beta \leq \alpha + \beta$. The other assertion follows by Induction on β.
4. Suppose $\gamma \geq \alpha$ and choose a β such that $\alpha + \beta \leq \gamma < \alpha + \beta'$. Then $\gamma = \alpha + \beta$.
5. By induction on β.

Proof. Define $F(\beta) = \alpha + \beta$ for a given α.

1. F continuous by definition. Note that $\alpha + \beta < (\alpha + \beta)' = \alpha + \beta'$ so that $\forall \beta : F(\beta) < F(\beta')$. This yields that F is normal.
2. This assertion follows from the normality of F.
3. Since F is normal we see $\beta \leq \alpha + \beta$. The other assertion follows by Induction on β.
4. Suppose $\gamma \geq \alpha$ and choose a β such that $\alpha + \beta \leq \gamma < \alpha + \beta'$. Then $\gamma = \alpha + \beta$.
5. By induction on β.
6. By induction on γ.

Proof. Define $F(\beta) = \alpha + \beta$ for a given α.

1. F continuous by definition. Note that $\alpha + \beta < (\alpha + \beta)' = \alpha + \beta'$ so that $\forall \beta : F(\beta) < F(\beta')$. This yields that F is normal.
2. This assertion follows from the normality of F.
3. Since F is normal we see $\beta \leq \alpha + \beta$. The other assertion follows by Induction on β.
4. Suppose $\gamma \geq \alpha$ and choose a β such that $\alpha + \beta \leq \gamma < \alpha + \beta'$. Then $\gamma = \alpha + \beta$.
5. By induction on β.
6. By induction on γ.
7. By induction on α.

Proof. Define $F(\beta) = \alpha + \beta$ for a given α.

1. F continuous by definition. Note that $\alpha + \beta < (\alpha + \beta)' = \alpha + \beta'$ so that $\forall \beta : F(\beta) < F(\beta')$. This yields that F is normal.
2. This assertion follows from the normality of F.
3. Since F is normal we see $\beta \leq \alpha + \beta$. The other assertion follows by Induction on β.
4. Suppose $\gamma \geq \alpha$ and choose a β such that $\alpha + \beta \leq \gamma < \alpha + \beta'$. Then $\gamma = \alpha + \beta$.
5. By induction on β.
6. By induction on γ.
7. By induction on α.
8. If $0 < k < \omega$, then $k + \omega = \sup\{k + n \mid n < \omega\} = \sup\{m \mid m < \omega\} = \omega < \omega' \leq \omega + k$.

The product of ordinals

The product of two ordinals is defined by transfinite recursion:

$$
\begin{aligned}
\alpha \cdot 0 &= 0 \\
\alpha \cdot \beta' &= \alpha \cdot \beta + \alpha \\
\alpha \cdot \lambda &= \sup\{\alpha \cdot \xi \mid \xi < \lambda\} \text{ if } \lambda \in \mathrm{Lim}.
\end{aligned}
$$

Lemma

1. If $\alpha > 0$ then the function $\beta \mapsto \alpha \cdot \beta$ is normal.

Lemma

1. If $\alpha > 0$ then the function $\beta \mapsto \alpha \cdot \beta$ is normal.
2. $\alpha_0 \leq \alpha_1 \implies \alpha_0 \cdot \beta \leq \alpha_1 \cdot \beta$.

Lemma

1. If $\alpha > 0$ then the function $\beta \mapsto \alpha \cdot \beta$ is normal.
2. $\alpha_0 \leq \alpha_1 \implies \alpha_0 \cdot \beta \leq \alpha_1 \cdot \beta$.
3. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

Lemma

1. If $\alpha > 0$ then the function $\beta \mapsto \alpha \cdot \beta$ is normal.
2. $\alpha_0 \leq \alpha_1 \implies \alpha_0 \cdot \beta \leq \alpha_1 \cdot \beta$.
3. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.
4. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

Lemma

1. If $\alpha > 0$ then the function $\beta \mapsto \alpha \cdot \beta$ is normal.
2. $\alpha_0 \leq \alpha_1 \implies \alpha_0 \cdot \beta \leq \alpha_1 \cdot \beta$.
3. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.
4. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
5. $\alpha \cdot 0 = 0 = 0 \cdot \alpha$.

Lemma

1. If $\alpha > 0$ then the function $\beta \mapsto \alpha \cdot \beta$ is normal.
2. $\alpha_0 \leq \alpha_1 \implies \alpha_0 \cdot \beta \leq \alpha_1 \cdot \beta$.
3. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.
4. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
5. $\alpha \cdot 0 = 0 = 0 \cdot \alpha$.
6. $\alpha, \beta < \omega \implies \alpha \cdot \beta = \beta \cdot \alpha$.

UNIVERSITEIT GENT

Lemma

1. If $\alpha > 0$ then the function $\beta \mapsto \alpha \cdot \beta$ is normal.
2. $\alpha_0 \leq \alpha_1 \implies \alpha_0 \cdot \beta \leq \alpha_1 \cdot \beta$.
3. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.
4. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
5. $\alpha \cdot 0 = 0 = 0 \cdot \alpha$.
6. $\alpha, \beta < \omega \implies \alpha \cdot \beta = \beta \cdot \alpha$.
7. $1 < k < \omega \implies k \cdot \omega = \omega < \omega \cdot k$.

Lemma

1. If $\alpha > 0$ then the function $\beta \mapsto \alpha \cdot \beta$ is normal.
2. $\alpha_0 \leq \alpha_1 \implies \alpha_0 \cdot \beta \leq \alpha_1 \cdot \beta$.
3. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.
4. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
5. $\alpha \cdot 0 = 0 = 0 \cdot \alpha$.
6. $\alpha, \beta < \omega \implies \alpha \cdot \beta = \beta \cdot \alpha$.
7. $1 < k < \omega \implies k \cdot \omega = \omega < \omega \cdot k$.

Proof. All proofs are similar to the proofs we have seen before for the sum of ordinals, except the distributivity property which is proved by induction on γ:

Proof. All proofs are similar to the proofs we have seen before for the sum of ordinals, except the distributivity property which is proved by induction on γ:

- $\gamma = 0$. Then $\alpha \cdot (\beta + 0) = \alpha \cdot \beta = \alpha \cdot \beta + \alpha \cdot 0$.

UNIVERSITEIT GENT

Proof. All proofs are similar to the proofs we have seen before for the sum of ordinals, except the distributivity property which is proved by induction on γ:

- $\gamma = 0$. Then $\alpha \cdot (\beta + 0) = \alpha \cdot \beta = \alpha \cdot \beta + \alpha \cdot 0$.
- $\gamma = \xi'$. Then $\alpha \cdot (\beta + \xi') = \alpha \cdot (\beta + \xi)' = \alpha \cdot (\beta + \xi) + \alpha$.

Proof. All proofs are similar to the proofs we have seen before for the sum of ordinals, except the distributivity property which is proved by induction on γ:

- $\gamma = 0$. Then $\alpha \cdot (\beta + 0) = \alpha \cdot \beta = \alpha \cdot \beta + \alpha \cdot 0$.
- $\gamma = \xi'$. Then $\alpha \cdot (\beta + \xi') = \alpha \cdot (\beta + \xi)' = \alpha \cdot (\beta + \xi) + \alpha$. By induction hypthesis this is equal to $(\alpha \cdot \beta + \alpha \cdot \xi) + \alpha = \alpha \cdot \beta + (\alpha \cdot \xi + \alpha) = \alpha \cdot \beta + \alpha \cdot \gamma$.

Proof. All proofs are similar to the proofs we have seen before for the sum of ordinals, except the distributivity property which is proved by induction on γ:

- $\gamma = 0$. Then $\alpha \cdot (\beta + 0) = \alpha \cdot \beta = \alpha \cdot \beta + \alpha \cdot 0$.
- $\gamma = \xi'$. Then $\alpha \cdot (\beta + \xi') = \alpha \cdot (\beta + \xi)' = \alpha \cdot (\beta + \xi) + \alpha$. By induction hypthesis this is equal to $(\alpha \cdot \beta + \alpha \cdot \xi) + \alpha = \alpha \cdot \beta + (\alpha \cdot \xi + \alpha) = \alpha \cdot \beta + \alpha \cdot \gamma$.
- $\gamma \in \mathrm{Lim}$. Then $\alpha \cdot (\beta + \gamma) = \alpha \cdot \sup\{\beta + \xi \mid \xi < \gamma\} = \sup\{\alpha \cdot (\beta + \xi) \mid \xi < \gamma\}$.

Proof. All proofs are similar to the proofs we have seen before for the sum of ordinals, except the distributivity property which is proved by induction on γ:

- $\gamma = 0$. Then $\alpha \cdot (\beta + 0) = \alpha \cdot \beta = \alpha \cdot \beta + \alpha \cdot 0$.
- $\gamma = \xi'$. Then $\alpha \cdot (\beta + \xi') = \alpha \cdot (\beta + \xi)' = \alpha \cdot (\beta + \xi) + \alpha$. By induction hypthesis this is equal to $(\alpha \cdot \beta + \alpha \cdot \xi) + \alpha = \alpha \cdot \beta + (\alpha \cdot \xi + \alpha) = \alpha \cdot \beta + \alpha \cdot \gamma$.
- $\gamma \in \mathrm{Lim}$. Then $\alpha \cdot (\beta + \gamma) = \alpha \cdot \sup\{\beta + \xi \mid \xi < \gamma\} = \sup\{\alpha \cdot (\beta + \xi) \mid \xi < \gamma\}$. By the induction hypothesis the last term equals $\sup\{\alpha \cdot \beta + \alpha \cdot \xi \mid \xi < \gamma\} = \alpha \cdot \beta + \sup\{\alpha \cdot \xi \mid \xi < \gamma\} = \alpha \cdot \beta + \alpha \cdot \gamma$.

UNIVERSITEIT GENT

The exponentiation of ordinals

The exponentiation of two ordinals is defined by the following transfinite recursion:

$$
\begin{aligned}
\alpha^{0} &= 1 \\
\alpha^{\beta'} &= \alpha^{\beta} \cdot \alpha \\
\alpha^{\lambda} &= \sup\{\alpha^{\xi} \mid 0 < \xi < \lambda\} \text{ if } \lambda \in \mathrm{Lim}.
\end{aligned}
$$

Lemma

1. The function $\beta \mapsto \alpha^\beta$ is normal if $\alpha \geq 2$.

Lemma

1. The function $\beta \mapsto \alpha^\beta$ is normal if $\alpha \geq 2$.
2. $\alpha \leq \gamma \implies \alpha^\beta \leq \gamma^\beta$.

Lemma

1. The function $\beta \mapsto \alpha^\beta$ is normal if $\alpha \geq 2$.
2. $\alpha \leq \gamma \implies \alpha^\beta \leq \gamma^\beta$.
3. $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$.

Lemma

1. The function $\beta \mapsto \alpha^\beta$ is normal if $\alpha \geq 2$.
2. $\alpha \leq \gamma \implies \alpha^\beta \leq \gamma^\beta$.
3. $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$.
4. $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$.

UNIVERSITEIT GENT

Lemma

1. The function $\beta \mapsto \alpha^\beta$ is normal if $\alpha \geq 2$.
2. $\alpha \leq \gamma \implies \alpha^\beta \leq \gamma^\beta$.
3. $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$.
4. $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$.
5. If $\beta > \beta_0 > \cdots > \beta_n$ and $\alpha > \delta_0, \ldots, \delta_n$ then $\alpha^\beta > \alpha^{\beta_0} \cdot \delta_0 + \cdots + \alpha^{\beta_n} \cdot \delta_n$.

UNIVERSITEIT GENT

Proof.
All proofs are routine. Assertion 4 is proved by induction on γ and assertion 5 is proved by induction on n. For the induction step argue as follows:

$$\begin{aligned} \alpha^{\beta} &\geq \alpha^{\beta_0} \cdot \alpha \\ &\geq \alpha^{\beta_0} \cdot (\delta_0 + 1) \\ &> \alpha^{\beta_0} \cdot \delta_0 + \cdots + \alpha^{\beta_n} \cdot \delta_n. \end{aligned}$$

Cantor's theorem

1. For all $\alpha \geq 2$ and $\gamma \geq 1$ there exist uniquely determined β, δ, γ_0 such that $0 < \delta < \alpha$, $\gamma_0 < \alpha^\beta$ and

$$\gamma = \alpha^\beta \cdot \delta + \gamma_0.$$

UNIVERSITEIT GENT

Cantor's theorem

1. For all $\alpha \geq 2$ and $\gamma \geq 1$ there exist uniquely determined β, δ, γ_0 such that $0 < \delta < \alpha$, $\gamma_0 < \alpha^\beta$ and

$$\gamma = \alpha^\beta \cdot \delta + \gamma_0.$$

2. For all $\alpha \geq 2$ and $\gamma \geq 1$ there exist uniquely determined n, $\beta_0 > \cdots > \beta_n$, $0 < \delta_0, \ldots, \delta_n < \alpha$ such that

$$\gamma = \alpha^{\beta_0} \cdot \delta_0 + \cdots + \alpha^{\beta_n} \cdot \delta_n.$$

Cantor's theorem

1. For all $\alpha \geq 2$ and $\gamma \geq 1$ there exist uniquely determined β, δ, γ_0 such that $0 < \delta < \alpha$, $\gamma_0 < \alpha^\beta$ and

$$\gamma = \alpha^\beta \cdot \delta + \gamma_0.$$

2. For all $\alpha \geq 2$ and $\gamma \geq 1$ there exist uniquely determined n, $\beta_0 > \cdots > \beta_n$, $0 < \delta_0, \ldots, \delta_n < \alpha$ such that

$$\gamma = \alpha^{\beta_0} \cdot \delta_0 + \cdots + \alpha^{\beta_n} \cdot \delta_n.$$

UNIVERSITEIT GENT

Proof of the first assertion. We first prove existence. Since $\beta \mapsto \alpha^\beta$ is normal there exists a β such that $\alpha^\beta \leq \gamma < \alpha^{\beta+1}$.

Proof of the first assertion. We first prove existence. Since $\beta \mapsto \alpha^\beta$ is normal there exists a β such that $\alpha^\beta \leq \gamma < \alpha^{\beta+1}$. Therefore there exists a δ such that $0 < \delta < \alpha$ and $\alpha^\beta \cdot \delta \leq \gamma < \alpha^\beta \cdot (\delta + 1)$ and so there exists a γ_0 such that $\gamma_0 < \alpha^\beta$ and $\alpha^\beta \cdot \delta + \gamma_0 \leq \gamma < \alpha^\beta \cdot \delta + \gamma_0 + 1$.

Cantor's theorem

We now prove uniqueness.

Cantor's theorem

We now prove uniqueness. Assume that $\gamma = \alpha^{\beta} \cdot \delta + \gamma_0 = \alpha^{\beta_1} \cdot \delta_1 + \gamma_1$ where $0 < \delta, \delta_1 < \alpha$ and $\gamma_0 < \alpha^{\beta}$, $\gamma_1 < \alpha^{\beta_1}$. Since $0 < \delta < \alpha$ and $\gamma_0 < \alpha^{\beta}$ we find $\alpha^{\beta} \leq \gamma < \alpha^{\beta+1}$.

Cantor's theorem

We now prove uniqueness. Assume that $\gamma = \alpha^\beta \cdot \delta + \gamma_0 = \alpha^{\beta_1} \cdot \delta_1 + \gamma_1$ where $0 < \delta, \delta_1 < \alpha$ and $\gamma_0 < \alpha^\beta$, $\gamma_1 < \alpha^{\beta_1}$. Since $0 < \delta < \alpha$ and $\gamma_0 < \alpha^\beta$ we find $\alpha^\beta \leq \gamma < \alpha^{\beta+1}$. Indeed,

$$\alpha^{\beta+1} = \alpha^\beta \cdot \alpha \geq \alpha^\beta(\delta + 1) = \alpha^\beta \cdot \delta + \alpha^\beta > \alpha^\beta + \gamma_0.$$

Similarly we find $\alpha^{\beta_1} \leq \gamma < \alpha^{\beta_1+1}$. Since exponentiation is normal we find $\beta = \beta_1$. So we see $\gamma = \alpha^\beta \cdot \delta + \gamma_0 = \alpha^\beta \cdot \delta_1 + \gamma_1$. This yields

$$\alpha^\beta \cdot \delta \leq \gamma < \alpha^\beta \cdot (\delta + 1)$$
$$\alpha^\beta \cdot \delta_1 \leq \gamma < \alpha^\beta \cdot (\delta_1 + 1)$$

hence $\delta = \delta_1$.

Cantor's theorem

We now prove uniqueness. Assume that $\gamma = \alpha^{\beta} \cdot \delta + \gamma_0 = \alpha^{\beta_1} \cdot \delta_1 + \gamma_1$ where $0 < \delta, \delta_1 < \alpha$ and $\gamma_0 < \alpha^{\beta}$, $\gamma_1 < \alpha^{\beta_1}$. Since $0 < \delta < \alpha$ and $\gamma_0 < \alpha^{\beta}$ we find $\alpha^{\beta} \leq \gamma < \alpha^{\beta+1}$. Indeed,

$$\alpha^{\beta+1} = \alpha^{\beta} \cdot \alpha \geq \alpha^{\beta}(\delta + 1) = \alpha^{\beta} \cdot \delta + \alpha^{\beta} > \alpha^{\beta} + \gamma_0.$$

Similarly we find $\alpha^{\beta_1} \leq \gamma < \alpha^{\beta_1+1}$. Since exponentiation is normal we find $\beta = \beta_1$. So we see $\gamma = \alpha^{\beta} \cdot \delta + \gamma_0 = \alpha^{\beta} \cdot \delta_1 + \gamma_1$.

Cantor's theorem

We now prove uniqueness. Assume that $\gamma = \alpha^{\beta} \cdot \delta + \gamma_0 = \alpha^{\beta_1} \cdot \delta_1 + \gamma_1$ where $0 < \delta, \delta_1 < \alpha$ and $\gamma_0 < \alpha^{\beta}$, $\gamma_1 < \alpha^{\beta_1}$. Since $0 < \delta < \alpha$ and $\gamma_0 < \alpha^{\beta}$ we find $\alpha^{\beta} \leq \gamma < \alpha^{\beta+1}$. Indeed,

$$\alpha^{\beta+1} = \alpha^{\beta} \cdot \alpha \geq \alpha^{\beta}(\delta + 1) = \alpha^{\beta} \cdot \delta + \alpha^{\beta} > \alpha^{\beta} + \gamma_0.$$

Similarly we find $\alpha^{\beta_1} \leq \gamma < \alpha^{\beta_1+1}$. Since exponentiation is normal we find $\beta = \beta_1$. So we see $\gamma = \alpha^{\beta} \cdot \delta + \gamma_0 = \alpha^{\beta} \cdot \delta_1 + \gamma_1$. This yields

$$\alpha^{\beta} \cdot \delta \leq \gamma < \alpha^{\beta} \cdot (\delta + 1)$$
$$\alpha^{\beta} \cdot \delta_1 \leq \gamma < \alpha^{\beta} \cdot (\delta_1 + 1)$$

hence $\delta = \delta_1$

We now arrive at $\gamma = \alpha^{\beta} \cdot \delta + \gamma_0 = \alpha^{\beta} \cdot \delta + \gamma_1$. Since the ordinal sum is normal in the second argument we find $\gamma_0 = \gamma_1$.

We now arrive at $\gamma = \alpha^{\beta} \cdot \delta + \gamma_0 = \alpha^{\beta} \cdot \delta + \gamma_1$. Since the ordinal sum is normal in the second argument we find $\gamma_0 = \gamma_1$.
Proof of the second assertion by induction on γ. The previous assertion yields $\gamma = \alpha^{\beta} \cdot \delta + \gamma_0$ with $0 < \delta < \alpha$.

We now arrive at $\gamma = \alpha^\beta \cdot \delta + \gamma_0 = \alpha^\beta \cdot \delta + \gamma_1$. Since the ordinal sum is normal in the second argument we find $\gamma_0 = \gamma_1$.
Proof of the second assertion by induction on γ. The previous assertion yields $\gamma = \alpha^\beta \cdot \delta + \gamma_0$ with $0 < \delta < \alpha$. Since $\gamma_0 < \gamma$ the induction hypothesis yields that $\gamma_0 = \alpha^{\beta_1} \cdot \delta_1 + \cdots + \alpha^{\beta_n} \cdot \delta_n$ so that $\gamma = \alpha^\beta \cdot \delta + \alpha^{\beta_1} \cdot \delta_1 + \cdots + \alpha^{\beta_n} \cdot \delta_n$. We find $\beta > \beta_1$ since $\alpha^\beta > \gamma_0$.

We write $\alpha =_{\mathrm{CNF}} \omega^{\alpha_0} k_0 + \cdots + \omega^{\alpha_n} k_n$ where $\alpha_0 > \cdots > \alpha_n$. We call this the Cantor normal form of α.
Note that the CNF is unique by Cantor's theorem.

The class AP of additive principal numbers is defined by

$$\alpha \in \mathrm{AP} \iff \alpha > 0 \wedge \forall \xi, \eta < \alpha : \xi + \eta < \alpha$$

It is easy to see that 1 is the first additive principal number. It is also easy to see that the other additive principal numbers are limit ordinals.

Lemma

1. $\alpha \mapsto \omega^\alpha$ is the ordering function of AP.

Lemma

1. $\alpha \mapsto \omega^\alpha$ is the ordering function of AP.
2. $\alpha \in \mathrm{AP} \iff \forall \xi < \alpha : \xi + \alpha = \alpha$.

Proof of the first assertion. By induction on α. Let $F(\alpha) = \omega^\alpha$. The we have to show that F is a surjective and order preserving function from On onto AP.

Proof of the first assertion. By induction on α. Let $F(\alpha) = \omega^\alpha$. The we have to show that F is a surjective and order preserving function from On onto AP. We know already that F is order preserving. We still have to show that $\mathrm{range} F = \mathrm{AP}$.

Proof of the first assertion. By induction on α. Let $F(\alpha) = \omega^\alpha$. The we have to show that F is a surjective and order preserving function from On onto AP. We know already that F is order preserving. We still have to show that $\mathrm{range} F = \mathrm{AP}$.
Suppose $\alpha = 0$. Then $\omega^\alpha = \omega^0 = 1 \in \mathrm{AP}$.

Proof of the first assertion. By induction on α. Let $F(\alpha) = \omega^\alpha$. The we have to show that F is a surjective and order preserving function from On onto AP. We know already that F is order preserving. We still have to show that $\mathrm{range} F = \mathrm{AP}$.
Suppose $\alpha = 0$. Then $\omega^\alpha = \omega^0 = 1 \in \mathrm{AP}$.
Suppose $\alpha = \beta + 1$ and let $\xi, \eta < \omega^{\beta+1} = \omega^\beta \omega$.

Proof of the first assertion. By induction on α. Let $F(\alpha) = \omega^\alpha$. The we have to show that F is a surjective and order preserving function from On onto AP. We know already that F is order preserving. We still have to show that $\mathrm{range} F = \mathrm{AP}$.
Suppose $\alpha = 0$. Then $\omega^\alpha = \omega^0 = 1 \in \mathrm{AP}$.
Suppose $\alpha = \beta + 1$ and let $\xi, \eta < \omega^{\beta+1} = \omega^\beta \omega$. Then there exist $m, n < \omega$ such that $\xi < \omega^\beta n$ and $\eta < \omega^\beta m$. Then $\xi + \eta < \omega^\beta n + \omega^\beta m = \omega^\beta (n + m)$.

Proof of the first assertion. By induction on α. Let $F(\alpha) = \omega^\alpha$. The we have to show that F is a surjective and order preserving function from On onto AP. We know already that F is order preserving. We still have to show that $\mathrm{range} F = \mathrm{AP}$.
Suppose $\alpha = 0$. Then $\omega^\alpha = \omega^0 = 1 \in \mathrm{AP}$.
Suppose $\alpha = \beta + 1$ and let $\xi, \eta < \omega^{\beta+1} = \omega^\beta \omega$. Then there exist $m, n < \omega$ such that $\xi < \omega^\beta n$ and $\eta < \omega^\beta m$. Then $\xi + \eta < \omega^\beta n + \omega^\beta m = \omega^\beta (n + m)$. Because $n + m < \omega$ we have $\omega^\beta (n + m) < \omega^\beta \omega = \omega^{\beta+1} = \omega^\alpha$. This yields $\omega^\alpha \in \mathrm{AP}$.

Proof of the first assertion. By induction on α. Let $F(\alpha) = \omega^\alpha$. The we have to show that F is a surjective and order preserving function from On onto AP. We know already that F is order preserving. We still have to show that $\mathrm{range} F = \mathrm{AP}$.

Suppose $\alpha = 0$. Then $\omega^\alpha = \omega^0 = 1 \in \mathrm{AP}$.

Suppose $\alpha = \beta + 1$ and let $\xi, \eta < \omega^{\beta+1} = \omega^\beta\omega$. Then there exist $m, n < \omega$ such that $\xi < \omega^\beta n$ and $\eta < \omega^\beta m$. Then $\xi + \eta < \omega^\beta n + \omega^\beta m = \omega^\beta(n + m)$. Because $n + m < \omega$ we have $\omega^\beta(n + m) < \omega^\beta\omega = \omega^{\beta+1} = \omega^\alpha$. This yields $\omega^\alpha \in \mathrm{AP}$.

Assume now that $\alpha \in \mathrm{Lim}$ and let $\xi, \eta < \omega^\alpha$.

Proof of the first assertion. By induction on α. Let $F(\alpha) = \omega^\alpha$. The we have to show that F is a surjective and order preserving function from On onto AP. We know already that F is order preserving. We still have to show that $\mathrm{range} F = \mathrm{AP}$.
Suppose $\alpha = 0$. Then $\omega^\alpha = \omega^0 = 1 \in \mathrm{AP}$.
Suppose $\alpha = \beta + 1$ and let $\xi, \eta < \omega^{\beta+1} = \omega^\beta \omega$. Then there exist $m, n < \omega$ such that $\xi < \omega^\beta n$ and $\eta < \omega^\beta m$. Then $\xi + \eta < \omega^\beta n + \omega^\beta m = \omega^\beta (n + m)$. Because $n + m < \omega$ we have $\omega^\beta (n + m) < \omega^\beta \omega = \omega^{\beta+1} = \omega^\alpha$. This yields $\omega^\alpha \in \mathrm{AP}$.
Assume now that $\alpha \in \mathrm{Lim}$ and let $\xi, \eta < \omega^\alpha$. Then there exist $\alpha_1, \alpha_2 < \alpha$ with $\xi < \omega^{\alpha_1}$ and $\eta < \omega^{\alpha_2}$.

Proof of the first assertion. By induction on α. Let $F(\alpha) = \omega^\alpha$. The we have to show that F is a surjective and order preserving function from On onto AP. We know already that F is order preserving. We still have to show that $\mathrm{range} F = \mathrm{AP}$.
Suppose $\alpha = 0$. Then $\omega^\alpha = \omega^0 = 1 \in \mathrm{AP}$.
Suppose $\alpha = \beta + 1$ and let $\xi, \eta < \omega^{\beta+1} = \omega^\beta \omega$. Then there exist $m, n < \omega$ such that $\xi < \omega^\beta n$ and $\eta < \omega^\beta m$. Then $\xi + \eta < \omega^\beta n + \omega^\beta m = \omega^\beta (n + m)$. Because $n + m < \omega$ we have $\omega^\beta (n + m) < \omega^\beta \omega = \omega^{\beta+1} = \omega^\alpha$. This yields $\omega^\alpha \in \mathrm{AP}$.
Assume now that $\alpha \in \mathrm{Lim}$ and let $\xi, \eta < \omega^\alpha$. Then there exist $\alpha_1, \alpha_2 < \alpha$ with $\xi < \omega^{\alpha_1}$ and $\eta < \omega^{\alpha_2}$. Then $\xi + \eta < \omega^{\alpha_1} + \omega^{\alpha_2} < \omega^{\max(\alpha_0, \alpha_1)+1} < \omega^\alpha$.

Proof of the first assertion. By induction on α. Let $F(\alpha) = \omega^\alpha$. The we have to show that F is a surjective and order preserving function from On onto AP. We know already that F is order preserving. We still have to show that $\mathrm{range} F = \mathrm{AP}$.
Suppose $\alpha = 0$. Then $\omega^\alpha = \omega^0 = 1 \in \mathrm{AP}$.
Suppose $\alpha = \beta + 1$ and let $\xi, \eta < \omega^{\beta+1} = \omega^\beta \omega$. Then there exist $m, n < \omega$ such that $\xi < \omega^\beta n$ and $\eta < \omega^\beta m$. Then $\xi + \eta < \omega^\beta n + \omega^\beta m = \omega^\beta (n + m)$. Because $n + m < \omega$ we have $\omega^\beta (n + m) < \omega^\beta \omega = \omega^{\beta+1} = \omega^\alpha$. This yields $\omega^\alpha \in \mathrm{AP}$.
Assume now that $\alpha \in \mathrm{Lim}$ and let $\xi, \eta < \omega^\alpha$. Then there exist $\alpha_1, \alpha_2 < \alpha$ with $\xi < \omega^{\alpha_1}$ and $\eta < \omega^{\alpha_2}$. Then $\xi + \eta < \omega^{\alpha_1} + \omega^{\alpha_2} < \omega^{\max(\alpha_0, \alpha_1)+1} < \omega^\alpha$. Hence $\omega^\alpha \in \mathrm{AP}$.

Finally suppose that $\alpha \in \mathrm{AP}$. Let $\alpha =_{\mathrm{CNF}} \omega^{\alpha_0} k_0 + \cdots + \omega^{\alpha_n} k_n$.

Finally suppose that $\alpha \in \mathrm{AP}$. Let $\alpha =_{\mathrm{CNF}} \omega^{\alpha_0} k_0 + \cdots + \omega^{\alpha_n} k_n$. If $n > 0$ or $n = 0$ and $k_0 > 1$ then
$\alpha = \omega^{\alpha_0} + \omega^{\alpha_0} \cdot (k_0 - 1) + \cdots + \omega^{\alpha_n} k_n$ would show that $\alpha \notin \mathrm{AP}$.

Proof of the second assertion.
Suppose $\alpha \in \mathrm{AP}$. Then there are two cases:

- $\alpha = 1$. This case is trivial because the only $\xi < \alpha$ is the ordinal 0 and in this case we have $0 + \alpha = \alpha$.

Proof of the second assertion.
Suppose $\alpha \in \mathrm{AP}$. Then there are two cases:

- $\alpha = 1$. This case is trivial because the only $\xi < \alpha$ is the ordinal 0 and in this case we have $0 + \alpha = \alpha$.
- $\alpha \in \mathrm{Lim}$. Suppose $\xi < \alpha$. Then $\xi + \alpha = \sup\{\xi + \eta \mid \eta < \alpha\} \leq \alpha$.

Proof of the second assertion.
Suppose $\alpha \in \mathrm{AP}$. Then there are two cases:

- $\alpha = 1$. This case is trivial because the only $\xi < \alpha$ is the ordinal 0 and in this case we have $0 + \alpha = \alpha$.
- $\alpha \in \mathrm{Lim}$. Suppose $\xi < \alpha$. Then $\xi + \alpha = \sup\{\xi + \eta \mid \eta < \alpha\} \leq \alpha$. We have $\xi + \alpha \geq \alpha$ and so $\xi + \alpha = \alpha$.

Proof of the second assertion.
Suppose $\alpha \in \mathrm{AP}$. Then there are two cases:

- $\alpha = 1$. This case is trivial because the only $\xi < \alpha$ is the ordinal 0 and in this case we have $0 + \alpha = \alpha$.
- $\alpha \in \mathrm{Lim}$. Suppose $\xi < \alpha$. Then $\xi + \alpha = \sup\{\xi + \eta \mid \eta < \alpha\} \leq \alpha$. We have $\xi + \alpha \geq \alpha$ and so $\xi + \alpha = \alpha$.

For the other direction, suppose $\xi + \alpha = \alpha$ for all $\xi < \alpha$.

Proof of the second assertion.
Suppose $\alpha \in \mathrm{AP}$. Then there are two cases:

- $\alpha = 1$. This case is trivial because the only $\xi < \alpha$ is the ordinal 0 and in this case we have $0 + \alpha = \alpha$.
- $\alpha \in \mathrm{Lim}$. Suppose $\xi < \alpha$. Then $\xi + \alpha = \sup\{\xi + \eta \mid \eta < \alpha\} \le \alpha$. We have $\xi + \alpha \ge \alpha$ and so $\xi + \alpha = \alpha$.

For the other direction, suppose $\xi + \alpha = \alpha$ for all $\xi < \alpha$. Suppose $\xi, \eta < \alpha$. Then $\xi + \alpha, \eta + \alpha = \alpha$ and thence $\xi + \eta < \xi + \alpha = \alpha$ so that $\alpha \in \mathrm{AP}$.

We write $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_n$ if $\alpha = \alpha_0 + \cdots + \alpha_n$ and $\alpha_0 \geq \cdots \geq \alpha_n$ and $\alpha_0, \ldots, \alpha_n \in \mathrm{AP}$.

Lemma

For every $\alpha > 0$ there exists uniquely determined ordinals $\alpha_0, \ldots, \alpha_n$ such that $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_n$.

We write $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_n$ if $\alpha = \alpha_0 + \cdots + \alpha_n$ and $\alpha_0 \geq \cdots \geq \alpha_n$ and $\alpha_0, \ldots, \alpha_n \in \mathrm{AP}$.

Lemma

For every $\alpha > 0$ there exists uniquely determined ordinals $\alpha_0, \ldots, \alpha_n$ such that $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_n$.

Proof. By looking at the Cantor normal form of α.

We write $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_n$ if $\alpha = \alpha_0 + \cdots + \alpha_n$ and $\alpha_0 \geq \cdots \geq \alpha_n$ and $\alpha_0, \ldots, \alpha_n \in \mathrm{AP}$.

Lemma

For every $\alpha > 0$ there exists uniquely determined ordinals $\alpha_0, \ldots, \alpha_n$ such that $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_n$.

Proof. By looking at the Cantor normal form of α.

The natural sum of ordinals

The natural sum $\alpha \oplus \beta$ is defined by

1. $\alpha \oplus 0 = \alpha = 0 \oplus \alpha$.
2. If $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_n$ and $\beta =_{\mathrm{NF}} \alpha_{n+1} + \cdots + \alpha_{n+m}$ then $\alpha \oplus \beta = \alpha_{p(0)} + \cdots + \alpha_{p(n+m)}$ where $p : \mathbb{N} \to \mathbb{N}$ is a bijection from $\{0, \ldots, n+m\} \to \{0, \ldots, n+m\}$ with $\alpha_{p(0)} \geq \cdots \geq \alpha_{p(n+m)}$.

Lemma

1. $\alpha \oplus \beta = \beta \oplus \alpha$.

Lemma

1. $\alpha \oplus \beta = \beta \oplus \alpha$.
2. $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$.

Lemma

1. $\alpha \oplus \beta = \beta \oplus \alpha$.
2. $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$.
3. If $\alpha_0, \ldots, \alpha_n \in \mathrm{AP}$ with $\alpha_0 \geq \cdots \geq \alpha_n$ then $\alpha_0 + \cdots + \alpha_n = \alpha_0 \oplus \cdots \oplus \alpha_n$.

Lemma

1. $\alpha \oplus \beta = \beta \oplus \alpha$.
2. $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$.
3. If $\alpha_0, \ldots, \alpha_n \in \mathrm{AP}$ with $\alpha_0 \geq \cdots \geq \alpha_n$ then $\alpha_0 + \cdots + \alpha_n = \alpha_0 \oplus \cdots \oplus \alpha_n$.
4. $\beta < \gamma \implies \alpha \oplus \beta < \alpha \oplus \gamma$.

Lemma

1. $\alpha \oplus \beta = \beta \oplus \alpha$.
2. $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$.
3. If $\alpha_0, \ldots, \alpha_n \in \mathrm{AP}$ with $\alpha_0 \geq \cdots \geq \alpha_n$ then $\alpha_0 + \cdots + \alpha_n = \alpha_0 \oplus \cdots \oplus \alpha_n$.
4. $\beta < \gamma \implies \alpha \oplus \beta < \alpha \oplus \gamma$.
5. $\alpha, \beta < \omega^\gamma \implies \alpha \oplus \beta < \omega^\gamma$.

UNIVERSITEIT GENT

Lemma

1. $\alpha \oplus \beta = \beta \oplus \alpha$.
2. $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$.
3. If $\alpha_0, \ldots, \alpha_n \in \mathrm{AP}$ with $\alpha_0 \geq \cdots \geq \alpha_n$ then $\alpha_0 + \cdots + \alpha_n = \alpha_0 \oplus \cdots \oplus \alpha_n$.
4. $\beta < \gamma \implies \alpha \oplus \beta < \alpha \oplus \gamma$.
5. $\alpha, \beta < \omega^\gamma \implies \alpha \oplus \beta < \omega^\gamma$.
6. $\alpha + \beta \leq \alpha \oplus \beta$.

One can interprete the natural sum of ordinals α en β as union of the multisets of their exponents.

For $\alpha \in \mathrm{On}$ we define functions $\varphi_\alpha : \mathrm{On} \to \mathrm{On}$ as follows.

1. $\varphi_0 = \mathrm{Enum}_{\mathrm{AP}}$.
2. $\varphi_{\alpha+1} = \mathrm{Enum}_{\{\beta\in\mathrm{On}:\beta=\varphi_\alpha\beta\}}$.
3. $\varphi_\lambda = \mathrm{Enum}_{\{\beta\in\mathrm{On}:(\forall\xi<\lambda)\beta=\varphi_\xi\beta\}}$.

$\varphi\alpha\beta := \varphi_\alpha\beta$.

Lemma

The function φ_α is normal for every $\alpha \in$ *On*.

Lemma

The function φ_α is normal for every $\alpha \in$ *On*.

Proof. By induction on α.

Proof. By induction on α.
Let $Cr(\alpha)$ be the range of φ_α. One shows that $Cr(\alpha)$ is club for all α.

Proof. By induction on α.
Let $Cr(\alpha)$ be the range of φ_α. One shows that $Cr(\alpha)$ is club for all α.
Closedness:

Proof. By induction on α.
Let $Cr(\alpha)$ be the range of φ_α. One shows that $Cr(\alpha)$ is club for all α.
Closedness: Let us consider the case that $\alpha \in Lim$.

Proof. By induction on α.
Let *Cr*(α) be the range of φ_α. One shows that *Cr*(α) is club for all α.
Closedness: Let us consider the case that $\alpha \in$ *Lim*. Let $A \subseteq Cr(\alpha) = \bigcap_{\xi<\alpha} Cr(\xi)$.

Proof. By induction on α.
Let $Cr(\alpha)$ be the range of φ_α. One shows that $Cr(\alpha)$ is club for all α.
Closedness: Let us consider the case that $\alpha \in Lim$. Let $A \subseteq Cr(\alpha) = \bigcap_{\xi<\alpha} Cr(\xi)$. Then $A \subseteq Cr(\xi)$ for all $\xi < \alpha$.

Proof. By induction on α.
Let $Cr(\alpha)$ be the range of φ_α. One shows that $Cr(\alpha)$ is club for all α.
Closedness: Let us consider the case that $\alpha \in Lim$. Let $A \subseteq Cr(\alpha) = \bigcap_{\xi<\alpha} Cr(\xi)$. Then $A \subseteq Cr(\xi)$ for all $\xi < \alpha$. Hence by i.h. $\sup A \in Cr(\xi)$ for all $\xi < \alpha$ so that $\sup A \in Cr(\alpha)$.

Proof. By induction on α.
Let $Cr(\alpha)$ be the range of φ_α. One shows that $Cr(\alpha)$ is club for all α.
Closedness: Let us consider the case that $\alpha \in Lim$. Let $A \subseteq Cr(\alpha) = \bigcap_{\xi<\alpha} Cr(\xi)$. Then $A \subseteq Cr(\xi)$ for all $\xi < \alpha$. Hence by i.h. $\sup A \in Cr(\xi)$ for all $\xi < \alpha$ so that $\sup A \in Cr(\alpha)$.
Unboundedness: Fix $\beta \in On$.

Proof. By induction on α.
Let $Cr(\alpha)$ be the range of φ_α. One shows that $Cr(\alpha)$ is club for all α.
Closedness: Let us consider the case that $\alpha \in Lim$. Let $A \subseteq Cr(\alpha) = \bigcap_{\xi<\alpha} Cr(\xi)$. Then $A \subseteq Cr(\xi)$ for all $\xi < \alpha$. Hence by i.h. $\sup A \in Cr(\xi)$ for all $\xi < \alpha$ so that $\sup A \in Cr(\alpha)$.
Unboundedness: Fix $\beta \in On$. Let $\gamma_0 > \alpha$.

Proof. By induction on α.
Let $Cr(\alpha)$ be the range of φ_α. One shows that $Cr(\alpha)$ is club for all α.
Closedness: Let us consider the case that $\alpha \in Lim$. Let $A \subseteq Cr(\alpha) = \bigcap_{\xi<\alpha} Cr(\xi)$. Then $A \subseteq Cr(\xi)$ for all $\xi < \alpha$. Hence by i.h. $\sup A \in Cr(\xi)$ for all $\xi < \alpha$ so that $\sup A \in Cr(\alpha)$.
Unboundedness: Fix $\beta \in On$. Let $\gamma_0 > \alpha$. By recursion let $\gamma_{n+1} := \sup\{\varphi_\xi \gamma_n : \xi < \alpha\}$. Let $\gamma = \sup \gamma_n$.

Proof. By induction on α.
Let $Cr(\alpha)$ be the range of φ_α. One shows that $Cr(\alpha)$ is club for all α.
Closedness: Let us consider the case that $\alpha \in Lim$. Let $A \subseteq Cr(\alpha) = \bigcap_{\xi<\alpha} Cr(\xi)$. Then $A \subseteq Cr(\xi)$ for all $\xi < \alpha$. Hence by i.h. $\sup A \in Cr(\xi)$ for all $\xi < \alpha$ so that $\sup A \in Cr(\alpha)$.
Unboundedness: Fix $\beta \in On$. Let $\gamma_0 > \alpha$. By recursion let $\gamma_{n+1} := \sup\{\varphi_\xi \gamma_n : \xi < \alpha\}$. Let $\gamma = \sup \gamma_n$. Then $\beta < \gamma$.
Now let $\xi < \alpha$.

Proof. By induction on α.
Let $Cr(\alpha)$ be the range of φ_α. One shows that $Cr(\alpha)$ is club for all α.
Closedness: Let us consider the case that $\alpha \in Lim$. Let $A \subseteq Cr(\alpha) = \bigcap_{\xi<\alpha} Cr(\xi)$. Then $A \subseteq Cr(\xi)$ for all $\xi < \alpha$. Hence by i.h. $\sup A \in Cr(\xi)$ for all $\xi < \alpha$ so that $\sup A \in Cr(\alpha)$.
Unboundedness: Fix $\beta \in On$. Let $\gamma_0 > \alpha$. By recursion let $\gamma_{n+1} := \sup\{\varphi_\xi\gamma_n : \xi < \alpha\}$. Let $\gamma = \sup \gamma_n$. Then $\beta < \gamma$.
Now let $\xi < \alpha$.
Then $\varphi_\xi\gamma = \sup \varphi_\xi\gamma_n \leq \sup \gamma_{n+1} = \gamma$.

Proof. By induction on α.
Let $Cr(\alpha)$ be the range of φ_α. One shows that $Cr(\alpha)$ is club for all α.
Closedness: Let us consider the case that $\alpha \in Lim$. Let $A \subseteq Cr(\alpha) = \bigcap_{\xi<\alpha} Cr(\xi)$. Then $A \subseteq Cr(\xi)$ for all $\xi < \alpha$. Hence by i.h. $\sup A \in Cr(\xi)$ for all $\xi < \alpha$ so that $\sup A \in Cr(\alpha)$.
Unboundedness: Fix $\beta \in On$. Let $\gamma_0 > \alpha$. By recursion let $\gamma_{n+1} := \sup\{\varphi_\xi \gamma_n : \xi < \alpha\}$. Let $\gamma = \sup \gamma_n$. Then $\beta < \gamma$.
Now let $\xi < \alpha$.
Then $\varphi_\xi \gamma = \sup \varphi_\xi \gamma_n \leq \sup \gamma_{n+1} = \gamma$. Hence $\gamma \in Cr(\alpha)$.

UNIVERSITEIT GENT

Lemma

$\varphi\alpha\beta = \varphi\gamma\delta$ iff

1. $\alpha < \gamma$ and $\beta = \varphi\gamma\delta$, or
2. $\alpha = \gamma$ and $\beta = \delta$, or
3. $\gamma < \alpha$ and $\varphi\alpha\beta = \delta$.

Lemma

$\varphi\alpha\beta = \varphi\gamma\delta$ iff

1. $\alpha < \gamma$ and $\beta = \varphi\gamma\delta$, or
2. $\alpha = \gamma$ and $\beta = \delta$, or
3. $\gamma < \alpha$ and $\varphi\alpha\beta = \delta$.

Proof. If $\alpha < \gamma$ then $\varphi\alpha(\varphi\gamma\delta) = \varphi\gamma\delta$.

Lemma

$\varphi\alpha\beta = \varphi\gamma\delta$ iff

1. $\alpha < \gamma$ and $\beta = \varphi\gamma\delta$, or
2. $\alpha = \gamma$ and $\beta = \delta$, or
3. $\gamma < \alpha$ and $\varphi\alpha\beta = \delta$.

Proof. If $\alpha < \gamma$ then $\varphi\alpha(\varphi\gamma\delta) = \varphi\gamma\delta$. Hence $\varphi\alpha\beta = \varphi\gamma\delta$ iff $\beta = \varphi\gamma\delta$.

Lemma

$\varphi\alpha\beta = \varphi\gamma\delta$ iff

1. $\alpha < \gamma$ and $\beta = \varphi\gamma\delta$, or
2. $\alpha = \gamma$ and $\beta = \delta$, or
3. $\gamma < \alpha$ and $\varphi\alpha\beta = \delta$.

Proof. If $\alpha < \gamma$ then $\varphi\alpha(\varphi\gamma\delta) = \varphi\gamma\delta$. Hence $\varphi\alpha\beta = \varphi\gamma\delta$ iff $\beta = \varphi\gamma\delta$. The case $\gamma < \alpha$ is similar.

Lemma

$\varphi\alpha\beta = \varphi\gamma\delta$ iff

1. $\alpha < \gamma$ and $\beta = \varphi\gamma\delta$, or
2. $\alpha = \gamma$ and $\beta = \delta$, or
3. $\gamma < \alpha$ and $\varphi\alpha\beta = \delta$.

Proof. If $\alpha < \gamma$ then $\varphi\alpha(\varphi\gamma\delta) = \varphi\gamma\delta$. Hence $\varphi\alpha\beta = \varphi\gamma\delta$ iff $\beta = \varphi\gamma\delta$. The case $\gamma < \alpha$ is similar. For $\alpha = \gamma$ the assertion is trivial.

Lemma

$\varphi\alpha\beta < \varphi\gamma\delta$ iff

1. $\alpha < \gamma$ and $\beta < \varphi\gamma\delta$, or
2. $\alpha = \gamma$ and $\beta < \delta$, or
3. $\gamma < \alpha$ and $\varphi\alpha\beta < \delta$.

Proof. If $\alpha < \gamma$ then $\varphi\alpha(\varphi\gamma\delta) = \varphi\gamma\delta$.

Lemma

$\varphi\alpha\beta < \varphi\gamma\delta$ iff

1. $\alpha < \gamma$ and $\beta < \varphi\gamma\delta$, or
2. $\alpha = \gamma$ and $\beta < \delta$, or
3. $\gamma < \alpha$ and $\varphi\alpha\beta < \delta$.

Proof. If $\alpha < \gamma$ then $\varphi\alpha(\varphi\gamma\delta) = \varphi\gamma\delta$. Hence $\varphi\alpha\beta < \varphi\gamma\delta$ iff $\beta < \varphi\gamma\delta$.

UNIVERSITEIT GENT

Lemma

$\varphi\alpha\beta < \varphi\gamma\delta$ iff

1. $\alpha < \gamma$ and $\beta < \varphi\gamma\delta$, or
2. $\alpha = \gamma$ and $\beta < \delta$, or
3. $\gamma < \alpha$ and $\varphi\alpha\beta < \delta$.

Proof. If $\alpha < \gamma$ then $\varphi\alpha(\varphi\gamma\delta) = \varphi\gamma\delta$. Hence $\varphi\alpha\beta < \varphi\gamma\delta$ iff $\beta < \varphi\gamma\delta$. The case $\gamma < \alpha$ is similar.

Lemma

$\varphi\alpha\beta < \varphi\gamma\delta$ iff

1. $\alpha < \gamma$ and $\beta < \varphi\gamma\delta$, or
2. $\alpha = \gamma$ and $\beta < \delta$, or
3. $\gamma < \alpha$ and $\varphi\alpha\beta < \delta$.

Proof. If $\alpha < \gamma$ then $\varphi\alpha(\varphi\gamma\delta) = \varphi\gamma\delta$. Hence $\varphi\alpha\beta < \varphi\gamma\delta$ iff $\beta < \varphi\gamma\delta$. The case $\gamma < \alpha$ is similar. For $\alpha = \gamma$ the assertion is trivial.

Lemma

For every $\gamma \in \mathrm{AP}$ there exist unique α and $\beta < \gamma$ such that $\gamma = \varphi\alpha\beta$.

Proof. Existence: By induction on α one shows $\alpha \leq \varphi\alpha 0$ (exercise).

Lemma

For every $\gamma \in \mathrm{AP}$ there exist unique α and $\beta < \gamma$ such that $\gamma = \varphi\alpha\beta$.

Proof. Existence: By induction on α one shows $\alpha \leq \varphi\alpha 0$ (exercise). Therefore $\gamma \leq \varphi\gamma 0 < \varphi\gamma\gamma$.

Lemma

For every $\gamma \in \mathrm{AP}$ there exist unique α and $\beta < \gamma$ such that $\gamma = \varphi\alpha\beta$.

Proof. Existence: By induction on α one shows $\alpha \leq \varphi\alpha 0$ (exercise). Therefore $\gamma \leq \varphi\gamma 0 < \varphi\gamma\gamma$. Let α be minimal such that $\gamma < \varphi\alpha\gamma$.

Lemma

For every $\gamma \in \mathrm{AP}$ there exist unique α and $\beta < \gamma$ such that $\gamma = \varphi\alpha\beta$.

Proof. Existence: By induction on α one shows $\alpha \leq \varphi\alpha 0$ (exercise). Therefore $\gamma \leq \varphi\gamma 0 < \varphi\gamma\gamma$. Let α be minimal such that $\gamma < \varphi\alpha\gamma$. If $\alpha > 0$ then $\varphi\xi\gamma = \gamma$ for all $\xi < \alpha$.

Lemma

For every $\gamma \in \mathrm{AP}$ there exist unique α and $\beta < \gamma$ such that $\gamma = \varphi\alpha\beta$.

Proof. Existence: By induction on α one shows $\alpha \leq \varphi\alpha 0$ (exercise). Therefore $\gamma \leq \varphi\gamma 0 < \varphi\gamma\gamma$. Let α be minimal such that $\gamma < \varphi\alpha\gamma$. If $\alpha > 0$ then $\varphi\xi\gamma = \gamma$ for all $\xi < \alpha$. Therefore in all cases $\gamma \in Cr\alpha$ and so there exists a β such that $\gamma = \varphi\alpha\beta$.

Lemma

For every $\gamma \in \mathrm{AP}$ there exist unique α and $\beta < \gamma$ such that $\gamma = \varphi\alpha\beta$.

Proof. Existence: By induction on α one shows $\alpha \leq \varphi\alpha 0$ (exercise). Therefore $\gamma \leq \varphi\gamma 0 < \varphi\gamma\gamma$. Let α be minimal such that $\gamma < \varphi\alpha\gamma$. If $\alpha > 0$ then $\varphi\xi\gamma = \gamma$ for all $\xi < \alpha$. Therefore in all cases $\gamma \in Cr\alpha$ and so there exists a β such that $\gamma = \varphi\alpha\beta$. Since $\gamma < \varphi\alpha\gamma$ we have $\beta < \gamma$.

Uniqueness.

Uniqueness. Assume $\gamma = \varphi\alpha\beta = \varphi\xi\delta$ and $\beta, \delta < \gamma$.

Uniqueness. Assume $\gamma = \varphi\alpha\beta = \varphi\xi\delta$ and $\beta, \delta < \gamma$. Then a previous Lemma yields $\alpha = \xi$ and $\beta = \eta$.

Lemma

There exists $\Gamma_0 := \min\{\alpha : \alpha = \varphi\alpha 0\}$.

Lemma

There exists $\Gamma_0 := \min\{\alpha : \alpha = \varphi\alpha 0\}$. $\{\alpha : \alpha = \varphi\alpha 0\}$ is a club.

Proof. Let $\gamma_0 := 0$ and $\gamma_{n+1} := \varphi_{\gamma_n} 0$.

Lemma

There exists $\Gamma_0 := \min\{\alpha : \alpha = \varphi\alpha 0\}$. $\{\alpha : \alpha = \varphi\alpha 0\}$ is a club.

Proof. Let $\gamma_0 := 0$ and $\gamma_{n+1} := \varphi_{\gamma_n} 0$. Let $\gamma := \sup \gamma_n$.

Lemma

There exists $\Gamma_0 := \min\{\alpha : \alpha = \varphi\alpha 0\}$. $\{\alpha : \alpha = \varphi\alpha 0\}$ is a club.

Proof. Let $\gamma_0 := 0$ and $\gamma_{n+1} := \varphi_{\gamma_n} 0$. Let $\gamma := \sup \gamma_n$. Then $\gamma = \varphi\gamma 0$. (exercise).

Fundamental sequences and the Hardy hierarchy

From now on we restrict ourselves to ordinals below $\varphi 10 = \varepsilon_0$. Let $\alpha[n]$ is the n-th element of the fundamental sequence for $\alpha \in \mathrm{Lim}$:

$$\alpha[n] = \begin{cases} 0 & \text{if } \alpha \in \{0,1\} \\ \alpha_0 + \cdots + \alpha_{m-1} + \alpha_m[n] & \text{if } \alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_m \end{cases}$$

$$\omega^{\alpha+1}[n] = \omega^{\alpha}(n+1)$$

$$\omega^{\lambda}[n] = \omega^{\lambda[n]} \text{ if } \lambda \in \mathrm{Lim}.$$

The Hardy hierarchy is defined as follows:

$$
\begin{aligned}
H_0(n) &= n \\
H_{\alpha+1}(n) &= H_\alpha(n+1) \\
H_\lambda(n) &= H_{\lambda[n]}(n+1) \qquad \text{where } \lambda \in \mathrm{Lim}
\end{aligned}
$$

Let $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ be in Cantor normal form. Then

$$N(\alpha) = n + N(\alpha_1) + \cdots + N(\alpha_n)$$

We say $\mathrm{NF}(\alpha, \beta)$ if one the following conditions hold:

1. $\alpha = 0$;
2. $\beta = 0$;
3. $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$ and $\alpha_1 \geq \beta_1$.

The Bachmann property

Lemma

1. $\alpha \in \mathrm{Lim} \implies \alpha[n] < \alpha[n+1]$ and $\alpha[n] \to \alpha$ if $n \to \omega$
2. $\alpha > 0 \implies N(\alpha[0]) < N(\alpha)$

Proof by induction on α.

Lemma

$\alpha[n] < \beta < \alpha \implies \alpha[n] \leq \beta[0].$

Lemma

$\alpha[n] < \beta < \alpha \implies \alpha[n] \leq \beta[0]$.

Proof. Assume that $\beta =_{\mathrm{NF}} \beta_0 + \cdots + \beta_k$ with $k \geq 0$. There are the following three cases.

Lemma

$\alpha[n] < \beta < \alpha \implies \alpha[n] \leq \beta[0]$.

Proof. Assume that $\beta =_{\mathrm{NF}} \beta_0 + \cdots + \beta_k$ with $k \geq 0$. There are the following three cases.

Case 1. $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_m$ with $m > 0$:

$$\alpha[n] = \alpha_0 + \cdots + \alpha_m[n] < \beta_0 + \cdots + \beta_k < \alpha_0 + \cdots + \alpha_m$$

Lemma

$\alpha[n] < \beta < \alpha \implies \alpha[n] \leq \beta[0]$.

Proof. Assume that $\beta =_{\mathrm{NF}} \beta_0 + \cdots + \beta_k$ with $k \geq 0$. There are the following three cases.

Case 1. $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_m$ with $m > 0$:

$$\alpha[n] = \alpha_0 + \cdots + \alpha_m[n] < \beta_0 + \cdots + \beta_k < \alpha_0 + \cdots + \alpha_m$$

This yields $k \geq m$ en $\alpha_i = \beta_i$ for all $i < m$ so that

$$\alpha_m[n] < \beta_m + \cdots + \beta_k < \alpha_m \implies \alpha_m[n] \leq \beta_m < \alpha_m.$$

Lemma

$\alpha[n] < \beta < \alpha \implies \alpha[n] \leq \beta[0]$.

Proof. Assume that $\beta =_{\mathrm{NF}} \beta_0 + \cdots + \beta_k$ with $k \geq 0$. There are the following three cases.

Case 1. $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_m$ with $m > 0$:

$$\alpha[n] = \alpha_0 + \cdots + \alpha_m[n] < \beta_0 + \cdots + \beta_k < \alpha_0 + \cdots + \alpha_m$$

This yields $k \geq m$ en $\alpha_i = \beta_i$ for all $i < m$ so that

$$\alpha_m[n] < \beta_m + \cdots + \beta_k < \alpha_m \implies \alpha_m[n] \leq \beta_m < \alpha_m.$$

If $k = m$ then $\alpha_m[n] < \beta_m < \alpha_m$.

Lemma

$\alpha[n] < \beta < \alpha \implies \alpha[n] \leq \beta[0]$.

Proof. Assume that $\beta =_{\mathrm{NF}} \beta_0 + \cdots + \beta_k$ with $k \geq 0$. There are the following three cases.

Case 1. $\alpha =_{\mathrm{NF}} \alpha_0 + \cdots + \alpha_m$ with $m > 0$:

$$\alpha[n] = \alpha_0 + \cdots + \alpha_m[n] < \beta_0 + \cdots + \beta_k < \alpha_0 + \cdots + \alpha_m$$

This yields $k \geq m$ en $\alpha_i = \beta_i$ for all $i < m$ so that

$$\alpha_m[n] < \beta_m + \cdots + \beta_k < \alpha_m \implies \alpha_m[n] \leq \beta_m < \alpha_m.$$

If $k = m$ then $\alpha_m[n] < \beta_m < \alpha_m$. The induction hypothesis yields $\alpha_m[n] \leq \beta_m[0] \leq \alpha_m$. If $k > m$ then $\beta_m + \cdots + \beta_k[0] \geq \beta_m$.

Case 2. Suppose now $\alpha = \omega^{\gamma+1}$.

Case 2. Suppose now $\alpha = \omega^{\gamma+1}$. Then

$$\alpha[n] = \omega^{\gamma}(n+1) < \beta < \omega^{\gamma+1}$$

Case 2. Suppose now $\alpha = \omega^{\gamma+1}$. Then

$$\alpha[n] = \omega^{\gamma}(n+1) < \beta < \omega^{\gamma+1}$$

This yields $\beta_0 = \cdots = \beta_n = \omega^{\gamma}$ and $\beta_{n+1} \neq 0$ for $k \geq n+1$ and thus

$$\omega^{\gamma}(n+1) \leq \beta_0 + \cdots + \beta_n + \cdots + \beta_k[0]$$

Case 3. Suppose $\alpha = \omega^\lambda$. Then

$$\alpha[n] = \omega^\lambda[n] = \omega^{\lambda[n]} < \beta < \omega^\lambda$$

Case 3. Suppose $\alpha = \omega^\lambda$. Then

$$\alpha[n] = \omega^\lambda[n] = \omega^{\lambda[n]} < \beta < \omega^\lambda$$

We have $\beta_0 = \omega^\gamma \implies \lambda[n] \leq \gamma$. If $k > 0$ then $\beta[0] \geq \beta_0 \geq \omega^{\lambda[n]}$.

Case 3. Suppose $\alpha = \omega^\lambda$. Then

$$\alpha[n] = \omega^\lambda[n] = \omega^{\lambda[n]} < \beta < \omega^\lambda$$

We have $\beta_0 = \omega^\gamma \implies \lambda[n] \leq \gamma$. If $k > 0$ then $\beta[0] \geq \beta_0 \geq \omega^{\lambda[n]}$.
If $k = 0$ then $\lambda[n] < \gamma < \lambda$.

Case 3. Suppose $\alpha = \omega^\lambda$. Then

$$\alpha[n] = \omega^\lambda[n] = \omega^{\lambda[n]} < \beta < \omega^\lambda$$

We have $\beta_0 = \omega^\gamma \implies \lambda[n] \leq \gamma$. If $k > 0$ then $\beta[0] \geq \beta_0 \geq \omega^{\lambda[n]}$. If $k = 0$ then $\lambda[n] < \gamma < \lambda$. The induction hypothesis yields $\lambda[n] \leq \gamma[0]$.

Case 3. Suppose $\alpha = \omega^\lambda$. Then

$$\alpha[n] = \omega^\lambda[n] = \omega^{\lambda[n]} < \beta < \omega^\lambda$$

We have $\beta_0 = \omega^\gamma \implies \lambda[n] \leq \gamma$. If $k > 0$ then $\beta[0] \geq \beta_0 \geq \omega^{\lambda[n]}$. If $k = 0$ then $\lambda[n] < \gamma < \lambda$. The induction hypothesis yields $\lambda[n] \leq \gamma[0]$. Thence

$$\omega^{\lambda[n]} \leq \omega^{\gamma[0]} = \beta[0].$$

Lemma

$\alpha[n] < \beta < \alpha \implies N(\alpha[n]) < N(\beta)$.

Proof. This follows from the previous two lemmas.

Lemma

$\alpha < \beta \implies \alpha \leq \beta[N(\alpha)]$.

Proof. We obtain

$$\beta \in \mathrm{Lim} \implies N(\beta[n]) < N(\beta[n+1]) \implies N(\alpha) \leq N(\beta[N(\alpha)])$$

Suppose $\beta[N(\alpha)] < \alpha < \beta$. This yields a contradiction.

Lemma

1. $H_\alpha(n) < H_\alpha(n+1)$
2. $\beta[m] < \alpha < \beta \implies H_{\beta[m]}(n+1) \leq H_\alpha(n)$
3. $\beta < \alpha \wedge N(\beta) \leq n \implies H_\beta(n) < H_\alpha(n)$

Lemma

1. $H_\alpha(n) < H_\alpha(n+1)$
2. $\beta[m] < \alpha < \beta \implies H_{\beta[m]}(n+1) \leq H_\alpha(n)$
3. $\beta < \alpha \wedge N(\beta) \leq n \implies H_\beta(n) < H_\alpha(n)$

Proof. The first two assertions are proved by simultaneous induction on α.

Lemma

1. $H_\alpha(n) < H_\alpha(n+1)$
2. $\beta[m] < \alpha < \beta \implies H_{\beta[m]}(n+1) \leq H_\alpha(n)$
3. $\beta < \alpha \wedge N(\beta) \leq n \implies H_\beta(n) < H_\alpha(n)$

Proof. The first two assertions are proved by simultaneous induction on α. The first assertion is clear for $\alpha = 0$ and follows from the i.h. when $\alpha = \beta + 1$. If $\alpha \in \mathrm{Lim}$ then the second assertion yields

$H_\alpha(n) = H_{\alpha[n]}(n+1) < H_{\alpha[n]}(n+2) < H_{\alpha[n+1]}(n+2) = H_\alpha(n+1)$.

For a proof of the second assertion note $\beta[m] \leq \alpha[n] < \beta$

$H_{\beta[m]}(n+1) \leq H_\alpha[n](n) < H_\alpha[n](n+1) = H_\alpha(n)$.

The third assertion follows by induction on λ. $\beta < \alpha \wedge N(\beta) \leq n$ yields $\beta < \alpha[n]$ and hence $H_\beta(n) \leq H_\alpha[n](n) < H_\alpha(n)$.
Crucial observation: Let $k \geq n$ be minimal such that $\alpha[n] \ldots [k-1] = 0$. Then

$$H_\alpha(n) = H_{\alpha[n]}(n+1) = H_{\alpha[n][n+1]}(n+2) = H_{\alpha[n]\ldots[k-1]}(k) = k$$

Lemma

1. $\mathrm{NF}(\alpha, \beta) \implies H_{\alpha+\beta}(n) = H_\alpha(H_\beta(n))$.

Lemma

1. $\mathrm{NF}(\alpha,\beta) \implies H_{\alpha+\beta}(n) = H_\alpha(H_\beta(n))$.
2. $H_{\omega^{\alpha+1}}(n) = H_{\omega^\alpha}^{n+1}(n+1)$ en $H_{\omega^\lambda}(n) = H_{\omega^{\lambda[n]}}(n+1)$.

Lemma

1. $\mathrm{NF}(\alpha, \beta) \implies H_{\alpha+\beta}(n) = H_\alpha(H_\beta(n))$.
2. $H_{\omega^{\alpha+1}}(n) = H_{\omega^\alpha}^{n+1}(n+1)$ en $H_{\omega^\lambda}(n) = H_{\omega^{\lambda[n]}}(n+1)$.
3. For all primitive recursive functions f exists a k such that for all $\vec{x}$ we have $f(\vec{x}) < H_{\omega^k}(\max \vec{x})$.

Lemma

1. $\mathrm{NF}(\alpha, \beta) \implies H_{\alpha+\beta}(n) = H_\alpha(H_\beta(n))$.
2. $H_{\omega^{\alpha+1}}(n) = H_{\omega^\alpha}^{n+1}(n+1)$ en $H_{\omega^\lambda}(n) = H_{\omega^{\lambda[n]}}(n+1)$.
3. For all primitive recursive functions f exists a k such that for all $\vec{x}$ we have $f(\vec{x}) < H_{\omega^k}(\max \vec{x})$.

Proof.

1. By induction on β.

Lemma

1. $\mathrm{NF}(\alpha, \beta) \implies H_{\alpha+\beta}(n) = H_\alpha(H_\beta(n))$.
2. $H_{\omega^{\alpha+1}}(n) = H_{\omega^\alpha}^{n+1}(n+1)$ en $H_{\omega^\lambda}(n) = H_{\omega^{\lambda[n]}}(n+1)$.
3. For all primitive recursive functions f exists a k such that for all $\vec{x}$ we have $f(\vec{x}) < H_{\omega^k}(\max \vec{x})$.

Proof.

1. By induction on β.
2. $H_{\omega^{\alpha+1}}(n) = H_{\omega^{\alpha+1}[n]}(n+1) = H_{\omega^\alpha(n+1)}(n+1) = H_{\omega^\alpha}^{n+1}(n+1)$.

Lemma

1. $\mathrm{NF}(\alpha, \beta) \implies H_{\alpha+\beta}(n) = H_\alpha(H_\beta(n))$.
2. $H_{\omega^{\alpha+1}}(n) = H_{\omega^\alpha}^{n+1}(n+1)$ en $H_{\omega^\lambda}(n) = H_{\omega^{\lambda[n]}}(n+1)$.
3. For all primitive recursive functions f exists a k such that for all $\vec{x}$ we have $f(\vec{x}) < H_{\omega^k}(\max \vec{x})$.

Proof.

1. By induction on β.
2. $H_{\omega^{\alpha+1}}(n) = H_{\omega^{\alpha+1}[n]}(n+1) = H_{\omega^\alpha(n+1)}(n+1) = H_{\omega^\alpha}^{n+1}(n+1)$.
3. This assertion follows from (2).

UNIVERSITEIT GENT

Lemma

1. $\beta < \alpha \wedge N(\beta) \leq n \implies H_\beta(n) < H_\alpha(n)$.

Lemma

1. $\beta < \alpha \wedge N(\beta) \leq n \implies H_\beta(n) < H_\alpha(n)$.
2. $\lambda > 0 \implies 1 + N(\lambda[0]) = N(\lambda)$.

Lemma

1. $\beta < \alpha \wedge N(\beta) \leq n \implies H_\beta(n) < H_\alpha(n)$.
2. $\lambda > 0 \implies 1 + N(\lambda[0]) = N(\lambda)$.

Proof of the first assertion. Suppose $\beta < \alpha$. The assertion is proved by induction on α. If $\alpha = 0$ then the assertion follows trivially. So suppose $\alpha > 0$.

Proof of the first assertion. Suppose $\beta < \alpha$. The assertion is proved by induction on α. If $\alpha = 0$ then the assertion follows trivially. So suppose $\alpha > 0$. We find

$$H_\beta(n) < H_\beta(n+1) \overset{\text{IH}}{\leq} H_{\alpha[n]}(n+1) = H_\alpha(n)$$

Proof of assertion two by induction on λ.

Proof of assertion two by induction on λ. If $\lambda = 0$ then the theorem is trivial. So suppose that $\lambda = \alpha + 1$. Then $\lambda[0] = \alpha$

$$1 + N(\lambda[0]) = 1 + N(\alpha) = N(\alpha + 1) = N(\lambda)$$

Proof of assertion two by induction on λ. If $\lambda = 0$ then the theorem is trivial. So suppose that $\lambda = \alpha + 1$. Then $\lambda[0] = \alpha$

$$1 + N(\lambda[0]) = 1 + N(\alpha) = N(\alpha + 1) = N(\lambda)$$

If $\lambda = \omega^{\alpha+1}$ then $\lambda[0] = \omega^{\alpha}$ and

$$1 + N(\lambda[0]) = 2 + N(\alpha) = 1 + N(\lambda)$$

Proof of assertion two by induction on λ. If $\lambda = 0$ then the theorem is trivial. So suppose that $\lambda = \alpha + 1$. Then $\lambda[0] = \alpha$

$$1 + N(\lambda[0]) = 1 + N(\alpha) = N(\alpha + 1) = N(\lambda)$$

If $\lambda = \omega^{\alpha+1}$ then $\lambda[0] = \omega^{\alpha}$ and

$$1 + N(\lambda[0]) = 2 + N(\alpha) = 1 + N(\lambda)$$

If $\lambda = \omega^{\alpha}$ where $\alpha \in \mathrm{Lim}$ then we have

$$1 + N(\lambda[0]) = 1 + N(\omega^{\alpha[0]}) = 2 + N(\alpha[0]) \stackrel{\mathrm{IH}}{=} 1 + N(\alpha) = 1 + N(\lambda)$$

Lemma

1. $H_\alpha(n) \leq H_{\omega^\alpha}(n)$.

Lemma

1. $H_\alpha(n) \leq H_{\omega^\alpha}(n)$.
2. $H_\alpha(n) \leq H_{\alpha\oplus\beta}(n)$.

Lemma

1. $H_\alpha(n) \leq H_{\omega^\alpha}(n)$.
2. $H_\alpha(n) \leq H_{\alpha \oplus \beta}(n)$.
3. $N(\alpha) \leq H_\alpha(n)$.

Lemma

1. $H_\alpha(n) \leq H_{\omega^\alpha}(n)$.
2. $H_\alpha(n) \leq H_{\alpha \oplus \beta}(n)$.
3. $N(\alpha) \leq H_\alpha(n)$.

Proof of the first assertion. If $\alpha = 0$ then $H_0(n) = n < H_1(n)$.

Proof of the first assertion. If $\alpha = 0$ then $H_0(n) = n < H_1(n)$. For $\alpha = \beta + 1$ we find

$$H_{\omega^\alpha}(n) = H_{\omega^\beta(n+1)}(n) = H_{\omega^\beta}^{n+1}(n+1) \overset{\text{IH}}{\geq} H_\beta(n+1) = H_\alpha(n).$$

Proof of the first assertion. If $\alpha = 0$ then $H_0(n) = n < H_1(n)$. For $\alpha = \beta + 1$ we find

$$H_{\omega^\alpha}(n) = H_{\omega^\beta(n+1)}(n) = H_{\omega^\beta}^{n+1}(n+1) \overset{\text{IH}}{\geq} H_\beta(n+1) = H_\alpha(n).$$

If α is a limit, then:

$$H_\alpha(n) = H_{\alpha[n]}(n+1) \leq H_{\omega^{\alpha[n]}}(n+1) = H_{\omega^\alpha}(n)$$

Proof of the first assertion. If $\alpha = 0$ then $H_0(n) = n < H_1(n)$. For $\alpha = \beta + 1$ we find

$$H_{\omega^\alpha}(n) = H_{\omega^\beta(n+1)}(n) = H_{\omega^\beta}^{n+1}(n+1) \overset{\text{IH}}{\geq} H_\beta(n+1) = H_\alpha(n).$$

If α is a limit, then:

$$H_\alpha(n) = H_{\alpha[n]}(n+1) \leq H_{\omega^{\alpha[n]}}(n+1) = H_{\omega^\alpha}(n)$$

Proof of the second assertion. Suppose $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$ and $\beta = \omega^{\alpha_{m+1}} + \cdots + \omega^{\alpha_{m+n}}$.

Proof of the first assertion. If $\alpha = 0$ then $H_0(n) = n < H_1(n)$. For $\alpha = \beta + 1$ we find

$$H_{\omega^\alpha}(n) = H_{\omega^\beta(n+1)}(n) = H^{n+1}_{\omega^\beta}(n+1) \overset{\text{IH}}{\geq} H_\beta(n+1) = H_\alpha(n).$$

If α is a limit, then:

$$H_\alpha(n) = H_{\alpha[n]}(n+1) \leq H_{\omega^{\alpha[n]}}(n+1) = H_{\omega^\alpha}(n)$$

Proof of the second assertion. Suppose $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$ and $\beta = \omega^{\alpha_{m+1}} + \cdots + \omega^{\alpha_{m+n}}$. Then

$$H_\alpha(n) = H_{\omega^{\alpha_1}}(\ldots H_{\omega^{\alpha_m}}(n)\ldots)$$
$$H_{\alpha\oplus\beta} = H_{\omega^{\alpha_{\pi(1)}}}(\ldots H_{\omega^{\alpha_{\pi(n+m)}}}(n)\ldots)$$

whereπ is an permutation so that $\alpha_{\pi(1)} \geq \cdots \geq \alpha_{\pi(n+m)}$. It is easy to see that the assertion follows.

Proof of the third assertion.

Proof of the third assertion. If $\alpha = 0$ then the assertion is clear. For successors $\alpha + 1$ we find

$$N(\alpha+1) = 1 + N((\alpha+1)[0]) \leq H_{(\alpha+1)[0]}(0) \leq H_{(\alpha+1)[0]}(1) \leq H_{\alpha+1}(0$$

Proof of the third assertion. If $\alpha = 0$ then the assertion is clear. For successors $\alpha + 1$ we find

$$N(\alpha+1) = 1 + N((\alpha+1)[0]) \leq H_{(\alpha+1)[0]}(0) \leq H_{(\alpha+1)[0]}(1) \leq H_{\alpha+1}(0$$

If α is a limit:

$$N(\alpha) = 1 + N(\alpha[0]) \leq 1 + H_{\alpha[0]}(0) \leq H_{\alpha[0]}(1) = H_{\alpha}(0) \leq H_{\alpha}(n)$$

UNIVERSITEIT GENT

Applications

Now we study majorization properties for the Hardy hierarchy.

$$\begin{aligned} F^{\alpha}(x) &= \max(\{F(x)+1\} \cup \{C(F^{\gamma}, F^{\delta})(x) : \\ &\quad \gamma, \delta < \alpha \wedge N(\gamma), N(\delta) \leq F(x)\}) \end{aligned}$$

Applications

Now we study majorization properties for the Hardy hierarchy.

$$\begin{aligned} F^{\alpha}(x) &= \max(\{F(x)+1\} \cup \{C(F^{\gamma}, F^{\delta})(x) : \\ &\quad \gamma, \delta < \alpha \wedge N(\gamma), N(\delta) \leq F(x)\}) \end{aligned}$$

Lemma

Suppose that F is weakly increasing and fulfilling $F(x) \geq x$.

1. $\alpha < \beta \implies F^{\alpha}(x) \leq F^{\beta}(x)$.

Lemma

Suppose that F is weakly increasing and fulfilling $F(x) \geq x$.

1. $\alpha < \beta \implies F^{\alpha}(x) \leq F^{\beta}(x)$.
2. $4k \leq H_{\omega 2}(k)$ and $8k \leq H_{\omega 3}(k)$ or more generally $2^i k \leq H_{\omega i}(k)$.

Lemma

Suppose that F is weakly increasing and fulfilling $F(x) \geq x$.

1. $\alpha < \beta \implies F^\alpha(x) \leq F^\beta(x)$.
2. $4k \leq H_{\omega 2}(k)$ and $8k \leq H_{\omega 3}(k)$ or more generally $2^i k \leq H_{\omega i}(k)$.
3. $F \leq H_\alpha \implies F^\beta(x) \leq H_{\omega^{\alpha \oplus \beta + 1} + 8}(x)$.

Lemma

Suppose that F is weakly increasing and fulfilling $F(x) \geq x$.

1. $\alpha < \beta \implies F^{\alpha}(x) \leq F^{\beta}(x)$.
2. $4k \leq H_{\omega 2}(k)$ and $8k \leq H_{\omega 3}(k)$ or more generally $2^i k \leq H_{\omega i}(k)$.
3. $F \leq H_{\alpha} \implies F^{\beta}(x) \leq H_{\omega^{\alpha \oplus \beta + 1} + 8}(x)$.

Proof of the first assertion. By induction on α. For $\alpha = 0$ we obtain

$$F^{\alpha}(x) = F(x) + 1 \leq F^{\beta}(x)$$

Proof of the first assertion. By induction on α. For $\alpha = 0$ we obtain

$$F^{\alpha}(x) = F(x) + 1 \leq F^{\beta}(x)$$

For $\alpha > 0$ we find

$$F^{\alpha}(x) = F(x) + 1$$

or

$$F^{\alpha}(x) = F^{\gamma}(F^{\delta}(x)) + F^{\gamma}(x) + F^{\delta}(x)$$

for $\gamma, \delta < \alpha$ with $N(\gamma), N(\delta) \leq F(x)$.

Proof of the second assertion. We compute:

$$H_{\omega}(k) = H_{\omega[k]}(k+1) = H_{k+1}(k+1) = H_k(k+2) = H_{k-1}(k+3) =$$

$$H_{\omega 2}(k) = H_{(\omega 2)[k]}(k+1) = H_{\omega+k+1}(k+1) = H_{\omega}(2k+2) = H_{\omega[2k+2]}($$

$$= H_{2k+3}(2k+3) = 4k+6$$

$$H_{\omega 3}(k) = H_{\omega 2+k+1}(k+1) = H_{\omega 2}(2k+2) \geq 4(2k+2) \geq 8k$$

Proof of the third assertion. By induction on β we prove for all $x \geq 8$,

$$F^{\beta}(x) \leq H_{\omega^{\alpha \oplus \beta + 1}}(x)$$

and this yields the assertion.

Proof of the third assertion. By induction on β we prove for all $x \geq 8$,

$$F^{\beta}(x) \leq H_{\omega^{\alpha \oplus \beta + 1}}(x)$$

and this yields the assertion. For $\beta = 0$ we obtain:

$$\begin{aligned} F^{0}(x) = F(x) + 1 \leq H_{\alpha}(x) + 1 \leq H_{\alpha+1}(x) \leq H_{\alpha \oplus \beta + 1}(x) \\ \leq H_{\omega^{\alpha \oplus \beta + 1}}(x) \end{aligned}$$

Proof of the third assertion. By induction on β we prove for all $x \geq 8$,

$$F^{\beta}(x) \leq H_{\omega^{\alpha \oplus \beta+1}}(x)$$

and this yields the assertion. For $\beta = 0$ we obtain:

$$\begin{aligned} F^{0}(x) &= F(x) + 1 \leq H_{\alpha}(x) + 1 \leq H_{\alpha+1}(x) \leq H_{\alpha \oplus \beta+1}(x) \\ &\leq H_{\omega^{\alpha \oplus \beta+1}}(x) \end{aligned}$$

For $\beta > 0$ we have

$$F^{\beta}(x) = F^{\gamma}(F^{\delta}(x)) + F^{\gamma}(x) + F^{\delta}(x)$$

for $\gamma, \delta < \beta$. Let $\xi = \max(\gamma, \delta)$. Then we obtain

$$F^{\beta}(x) \leq F^{\xi}(F^{\xi}(x)) \cdot 3 \leq F^{\xi}(F^{\xi}(x)) \cdot 4$$

UNIVERSITEIT GENT

The induction hypothesis and (2) yield

$$
\begin{aligned}
F^{\xi}(F^{\xi}(x)) \cdot 4 &\leq H_{\omega 2}(H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(x))) \\
&\leq H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(x))))
\end{aligned}
$$

The induction hypothesis and (2) yield

$$F^{\xi}(F^{\xi}(x)) \cdot 4 \leq H_{\omega 2}(H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(x))) \\ \leq H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(x))))$$

Because of $H_{\alpha}(H_{\beta}(x)) = H_{\alpha+\beta}(x)$ for $NF(\alpha, \beta)$ we see

$$F^{\xi}(F^{\xi}(x)) \cdot 4 \leq H_{\omega^{\alpha \oplus \xi} \cdot 4}$$

The induction hypothesis and (2) yield

$$\begin{aligned} F^{\xi}(F^{\xi}(x)) \cdot 4 &\leq H_{\omega 2}(H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(x))) \\ &\leq H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(x)))) \end{aligned}$$

Because of $H_{\alpha}(H_{\beta}(x)) = H_{\alpha+\beta}(x)$ for $NF(\alpha, \beta)$ we see

$$F^{\xi}(F^{\xi}(x)) \cdot 4 \leq H_{\omega^{\alpha \oplus \xi} \cdot 4}$$

Moreover we find

$$\begin{aligned} N(\omega^{\alpha \oplus \xi+1}) &= 4(1 + N(\alpha) \oplus N(\xi) + 1) \leq 8(2H_{\alpha}(x)) = 16H_{\alpha}(x) \\ &\leq H_{\omega 4}(H_{\alpha}(x)) \leq H_{\omega^{\alpha \oplus \beta} 5}(x) \end{aligned}$$

Thus $F^{\beta}(x) \leq H_{\omega^{\alpha \oplus \xi+1} 4}(H_{\omega^{\alpha \oplus \beta} 5}(x))$.

The induction hypothesis and (2) yield

$$\begin{aligned} F^{\xi}(F^{\xi}(x)) \cdot 4 &\leq H_{\omega 2}(H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(x))) \\ &\leq H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(H_{\omega^{\alpha \oplus \xi+1}}(x)))) \end{aligned}$$

Because of $H_{\alpha}(H_{\beta}(x)) = H_{\alpha+\beta}(x)$ for $NF(\alpha, \beta)$ we see

$$F^{\xi}(F^{\xi}(x)) \cdot 4 \leq H_{\omega^{\alpha \oplus \xi} \cdot 4}$$

Moreover we find

$$\begin{aligned} N(\omega^{\alpha \oplus \xi+1}) = 4(1 + N(\alpha) \oplus N(\xi) + 1) &\leq 8(2H_{\alpha}(x)) = 16H_{\alpha}(x) \\ &\leq H_{\omega 4}(H_{\alpha}(x)) \leq H_{\omega^{\alpha \oplus \beta} 5}(x) \end{aligned}$$

Thus $F^{\beta}(x) \leq H_{\omega^{\alpha \oplus \xi+1} 4}(H_{\omega^{\alpha \oplus \beta} 5}(x))$.

Now we can show that $F^{\beta}(x) \leq H_{\omega^{\alpha \oplus \beta} 4}(H_{\omega^{\alpha \oplus \beta} 5}(x))$. For $\xi < \beta$ we have $\xi + 1 \leq \beta$ and there are two options:

- If $\xi + 1 = \beta$ then $H_{\omega^{\alpha \oplus \xi + 1} 4}(H_{\omega^{\alpha \oplus \beta} 5}(x)) = H_{\omega^{\alpha \oplus \beta} 4}(H_{\omega^{\alpha \oplus \beta} 5}(x))$.

Now we can show that $F^\beta(x) \le H_{\omega^{\alpha\oplus\beta}4}(H_{\omega^{\alpha\oplus\beta}5}(x))$. For $\xi < \beta$ we have $\xi + 1 \le \beta$ and there are two options:

- If $\xi + 1 = \beta$ then $H_{\omega^{\alpha\oplus\xi+1}4}(H_{\omega^{\alpha\oplus\beta}5}(x)) = H_{\omega^{\alpha\oplus\beta}4}(H_{\omega^{\alpha\oplus\beta}5}(x))$.
- If $\xi + 1 < \beta$ then $H_{\omega^{\alpha\oplus\xi+1}4}(H_{\omega^{\alpha\oplus\beta}5}(x)) = H_{\omega^{\alpha\oplus\beta}4}(H_{\omega^{\alpha\oplus\beta}5}(x))$ since the norm of the leftmost ordinal is controlled by the argument.

Now we can show that $F^\beta(x) \leq H_{\omega^{\alpha \oplus \beta}4}(H_{\omega^{\alpha \oplus \beta}5}(x))$. For $\xi < \beta$ we have $\xi + 1 \leq \beta$ and there are two options:

- If $\xi + 1 = \beta$ then $H_{\omega^{\alpha \oplus \xi+1}4}(H_{\omega^{\alpha \oplus \beta}5}(x)) = H_{\omega^{\alpha \oplus \beta}4}(H_{\omega^{\alpha \oplus \beta}5}(x))$.
- If $\xi + 1 < \beta$ then $H_{\omega^{\alpha \oplus \xi+1}4}(H_{\omega^{\alpha \oplus \beta}5}(x)) = H_{\omega^{\alpha \oplus \beta}4}(H_{\omega^{\alpha \oplus \beta}5}(x))$ since the norm of the leftmost ordinal is controlled by the argument. Finally we see

$$\begin{aligned} F^\beta(x) &\leq H_{\omega^{\alpha \oplus \beta}9}(x) \leq H_{\omega^{\alpha \oplus \beta+1}}(x) \leq H_{\omega^{\alpha \oplus \beta+1}}(x+8) \\ &\leq H_{\omega^{\alpha \oplus \beta+1}+1}(x+7) \leq \cdots \leq H_{\omega^{\alpha \oplus \beta+1}+8}(x) \end{aligned}$$

The Goodstein sequences

Let $m[k \leftarrow k+1]$ be the result of first writing m hereditarily in base k normal form and then second replacing base k by $k+1$.

The Goodstein sequences

Let $m[k \leftarrow k+1]$ be the result of first writing m hereditarily in base k normal form and then second replacing base k by $k+1$. Let $m_0 := m$ and $m_{k+1} := m_k[k+2 \leftarrow k+3] - 1$.

The Goodstein sequences

Let $m[k \leftarrow k+1]$ be the result of first writing m hereditarily in base k normal form and then second replacing base k by $k+1$.
Let $m_0 := m$ and $m_{k+1} := m_k[k+2 \leftarrow k+3] - 1$.
Then the assertion $\forall m \exists k m_k = 0$ is true but not provable in first order Peano arithmetic.

$$P_x\alpha = \begin{cases} 0 & \text{if } \alpha = 0 \\ \beta & \text{if } \alpha = \beta + 1 \\ P_x(\lambda[x]) & \text{if } \lambda \in Lim. \end{cases}$$

The modified Hardy hierarchy is defined as follows:

$$
\begin{aligned}
h_0(n) &= n \\
h_{\alpha+1}(n) &= h_\alpha(n+1) \\
h_\lambda(n) &= h_{\lambda[n]}(n) && \text{where } \lambda \in \mathrm{Lim}
\end{aligned}
$$

The slow growing hierarchy is defined as follows:

$$G_0(n) = 0$$
$$G_{\alpha+1}(n) = 1 + G_\alpha(n)$$
$$G_\lambda(n) = G_{\lambda[n]}(n) \qquad \text{where } \lambda \in \mathrm{Lim}$$

Lemma

1. If $\alpha > 0$ then $H_\alpha(x) = H_{P_x\alpha}(x+1)$.
2. Let $k \geq n$ be minimal such that $P_k \ldots P_{n+1} P_n \alpha = 0$. Then

$$h_\alpha(n) = k$$

Lemma

$H_\alpha(x) \leq h_\alpha(x+1) \leq H_\alpha(x+1)$.

Lemma

Write $G_x\alpha = G_\alpha(x)$. Then $P_x G_x \alpha = G_x P_x \alpha$.

Lemma

If $NF(\omega^\alpha, \beta)$ then $G_x(\omega^\alpha + \beta) = (x+1)^{G_x\alpha} + G_x\beta$.

Lemma

The termination of the Goodstein sequences follows from the totality of the function $x \mapsto H_{\varepsilon_0}(x)$.

Proof by putting the last lemmata together.

Lemma

The termination of the Goodstein sequences follows from the totality of the function $x \mapsto H_{\varepsilon_0}(x)$.

Proof by putting the last lemmata together. Let m be given.

Lemma

The termination of the Goodstein sequences follows from the totality of the function $x \mapsto H_{\varepsilon_0}(x)$.

Proof by putting the last lemmata together. Let m be given. Write m in base 2 representation.

Lemma

The termination of the Goodstein sequences follows from the totality of the function $x \mapsto H_{\varepsilon_0}(x)$.

Proof by putting the last lemmata together. Let m be given. Write m in base 2 representation. Let α be the result of replacing in this representation 2 by ω.

Lemma

The termination of the Goodstein sequences follows from the totality of the function $x \mapsto H_{\varepsilon_0}(x)$.

Proof by putting the last lemmata together. Let m be given. Write m in base 2 representation. Let α be the result of replacing in this representation 2 by ω. Then $m = G_1(\alpha)$.

Lemma

The termination of the Goodstein sequences follows from the totality of the function $x \mapsto H_{\varepsilon_0}(x)$.

Proof by putting the last lemmata together. Let m be given. Write m in base 2 representation. Let α be the result of replacing in this representation 2 by ω. Then $m = G_1(\alpha)$. Then $m_1 = G_2(\alpha) - 1 = P_2 G_2(\alpha) = G_2 P_2 \alpha$ and $m_2 = G_3 P_2 \alpha - 1 = G_3 P_3 P_2 \alpha$.

Lemma

The termination of the Goodstein sequences follows from the totality of the function $x \mapsto H_{\varepsilon_0}(x)$.

Proof by putting the last lemmata together. Let m be given. Write m in base 2 representation. Let α be the result of replacing in this representation 2 by ω. Then $m = G_1(\alpha)$. Then
$m_1 = G_2(\alpha) - 1 = P_2 G_2(\alpha) = G_2 P_2 \alpha$ and
$m_2 = G_3 P_2 \alpha - 1 = G_3 P_3 P_2 \alpha$. The $m_k = 0$ iff $P_{k+1} \ldots P_3 P_2 \alpha = 0$.

Lemma

The termination of the Goodstein sequences follows from the totality of the function $x \mapsto H_{\varepsilon_0}(x)$.

Proof by putting the last lemmata together. Let m be given. Write m in base 2 representation. Let α be the result of replacing in this representation 2 by ω. Then $m = G_1(\alpha)$. Then $m_1 = G_2(\alpha) - 1 = P_2 G_2(\alpha) = G_2 P_2 \alpha$ and $m_2 = G_3 P_2 \alpha - 1 = G_3 P_3 P_2 \alpha$. The $m_k = 0$ iff $P_{k+1} \ldots P_3 P_2 \alpha = 0$. This k corresponds to $h_\alpha(k)$ so essentially to $H_\alpha(k)$.

Thanks - Personal Information

Thank you for listening. The results of this talk will be covered next term in a lecture in Ghent. Interested (master,PhD) students or others are welcome.

Andreas Weiermann
Andreas.Weiermann@UGent.be

Department Mathematics
Ghent University
Krijgslaan 281
Building S22
9000 Ghent
Belgium