Modal Quantifiers, Potential Infinity, and Yablo sequences

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Outline

Yablo's paradox

Arithmetization of Yablo sentences

Potentially infinite domains and sl-semantics

Modal interpretation of quantifiers in potentially infinite domains

LYD makes YS *msl*-fail

Summing up

Yablo's paradox

Y_0	For any $k > 0$, Y_k is false.
Y_1	For any $k > 1$, Y_k is false.
Y_2	For any $k > 2$, Y_k is false.
	$\vdots$
Y_n	For any $k > n$, Y_k is false.
	$\vdots$

Yablo's paradox

$$\begin{array}{ll} Y_0 & \text{For any } k > 0,\ Y_k \text{ is false.} \\ Y_1 & \text{For any } k > 1,\ Y_k \text{ is false.} \\ Y_2 & \text{For any } k > 2,\ Y_k \text{ is false.} \\ & \vdots \\ Y_n & \text{For any } k > n,\ Y_k \text{ is false.} \\ & \vdots \end{array}$$

- Suppose Y_n.

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- Suppose Y_n.
- So for any $j > n$, $\neg Y_j$.

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\end{array}$$

- Suppose Y_n.
- So for any $j > n, \neg Y_j$.
- So $\neg Y_{n+1}$ and for any $j > n+1, \neg Y_j$.

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- Suppose Y_n.
- So for any $j > n$, $\neg Y_j$.
- So $\neg Y_{n+1}$ and for any $j > n+1$, $\neg Y_j$.
- So Y_{n+1}. Contradiction.

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- Suppose Y_n.
- So for any $j > n$, $\neg Y_j$.
- So $\neg Y_{n+1}$ and for any $j > n+1$, $\neg Y_j$.
- So Y_{n+1}. Contradiction.
- So $\neg Y_n$ unconditionally.

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- Suppose Y_n.
- So for any $j > n$, $\neg Y_j$.
- So $\neg Y_{n+1}$ and for any $j > n+1$, $\neg Y_j$.
- So Y_{n+1}. Contradiction.
- So $\neg Y_n$ unconditionally.
- So $\exists k > n\ Y_k$. Rinse and repeat.

Background assumptions (Ketland, 2005)

Uniform disquotation

$\forall x \; (Y(x) \equiv Tr(Y(x)))$

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For any particular n, assume $Y(\bar{n}) \equiv Tr(Y(\bar{n}))$.

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For any particular n, assume $Y(\bar{n}) \equiv Tr(Y(\bar{n}))$.

ω-rule

If for any n $\varphi(\bar{n})$, derive $\forall x\, \varphi(x)$.

Finitistic way out?

The idea

If the world is finite, there are only finitely many Yablo sentences, and the last one is vacuously true.

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Make sense of arithmetic in a formal finitistic setting.

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The challenge

Make sense of arithmetic in a formal finitistic setting.

The strategy

There **could** be more things: **potential infinity**.

Arithmetization of Yablo sentences

Theorem

Let T *be a first-order theory in the language* $\mathcal{L}_{Tr}$*, containing* Q *(nice). Then, for any* $\mathcal{L}_{Tr}$*-formula* $\varphi(x,y)$ *there is a* $\mathcal{L}_{Tr}$*-formula* $\psi(x)$ *such that:*

$$\mathrm{T} \vdash \psi(x) \equiv \varphi(x, \overline{\ulcorner \psi(x) \urcorner}).$$

Notation ("quantification over numerals")

$Qx\, P(\ulcorner \varphi(\dot{x}) \urcorner)$, where $Q \in \{\forall, \exists\}$, reads:

For all natural numbers x (there exists a natural number x such that), the result of substituting a numeral denoting x for a variable free in φ *has property* P*.*

Definition (Yablo formula/sentence)

$Y(x)$ is a Yablo formula in T iff

$$\mathrm{T} \vdash \forall x (Y(x) \equiv \forall w > x \neg Tr(\overline{\ulcorner Y(\dot{w}) \urcorner})).$$

Yablo sentences are of the form $Y(\bar{n})$.

Existence of Yablo Formulas (Priest, 1997)

Theorem

If T *is nice, there exists a Yablo formula in* T.

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Proof.

- Let $\varphi(x,y) = \forall w > x \, \neg Tr(sub(y, \ulcorner y \urcorner, name(w)))$.
- By the Diagonal Lemma, there is a formula $Y(x)$ s.t.:

$$\mathrm{T} \vdash Y(x) \equiv \forall w > x \, \neg Tr(sub(\overline{\ulcorner Y(x) \urcorner}, \ulcorner y \urcorner, name(w))).$$

- $\mathrm{T} \vdash Y(x) \equiv \forall w > x \, \neg Tr(\overline{\ulcorner Y(\dot{w}) \urcorner})$.

$\square$

ω-inconsistency of Yablo formulas (Ketland, 2005)

Statement

Definition (ω-consistency)

T is ω-consistent iff there is no $\varphi(x)$ s.t. simultaneously:

$$\forall n \in \omega \ \mathsf{T} \vdash \neg\varphi(\overline{n})$$

$$\mathsf{T} \vdash \exists x \varphi(x)$$

T is ω-inconsistent o/w.

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Definition ($\mathrm{PA_F}$)

Let $\mathcal{L}_F$ be standard language extended with F.

$$\mathrm{PA_F} := \mathrm{PA} \cup \{F(\overline{n}) \equiv \forall x > \overline{n} \, \neg F(x) \ : n \in \omega\}$$

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Theorem

$\mathrm{PA_F}$ *is ω-inconsistent.*

ω-inconsistency of Yablo formulas (Ketland, 2005)

Proof

- Work in PA_F. Fix an $n \in \omega$ and assume $F(\overline{n})$.

$$\forall x > \overline{n}\, \neg F(x). \tag{$\star$}$$

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Proof

- Work in $\mathsf{PA_F}$. Fix an $n \in \omega$ and assume $F(\overline{n})$.
$$\forall x > \overline{n}\, \neg F(x). \tag{$\star$}$$
- In particular, $\forall x > \overline{n+1}\, \neg F(x)$.

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- But from ($\star$), $\neg F(\overline{n+1})$ follows. Contradiction.

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- So unconditionally $\neg F(\overline{n})$:

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- But from ($\star$), $\neg F(\overline{n+1})$ follows. Contradiction.
- So unconditionally $\neg F(\overline{n})$:
$$\forall n \in \omega \quad \mathsf{PA}_\mathsf{F} \vdash \neg F(\overline{n}). \tag{1}$$

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- In particular, $\forall x > \overline{n+1}\, \neg F(x)$.
- This is equivalent to $F(\overline{n+1})$.
- But from ($\star$), $\neg F(\overline{n+1})$ follows. Contradiction.
- So unconditionally $\neg F(\overline{n})$:
$$\forall n \in \omega \quad PA_F \vdash \neg F(\overline{n}). \tag{1}$$
- By definition of PA_F:
$$\forall n \in \omega \quad PA_F \vdash \exists x > \overline{n}\, F(x).$$

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Proof

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- So unconditionally $\neg F(\overline{n})$:

$$\forall n \in \omega \quad \mathsf{PA_F} \vdash \neg F(\overline{n}). \tag{1}$$

- By definition of $\mathsf{PA_F}$:

$$\forall n \in \omega \quad \mathsf{PA_F} \vdash \exists x > \overline{n}\, F(x).$$

- In particular:

$$\mathsf{PA_F} \vdash \exists x\, F(x). \tag{2}$$

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Proof

- Work in $\mathrm{PA_F}$. Fix an $n \in \omega$ and assume $F(\overline{n})$.
$$\forall x > \overline{n}\, \neg F(x). \tag{$\star$}$$
- In particular, $\forall x > \overline{n+1}\, \neg F(x)$.
- This is equivalent to $F(\overline{n+1})$.
- But from ($\star$), $\neg F(\overline{n+1})$ follows. Contradiction.
- So unconditionally $\neg F(\overline{n})$:
$$\forall n \in \omega \;\; \mathrm{PA_F} \vdash \neg F(\overline{n}). \tag{1}$$
- By definition of $\mathrm{PA_F}$:
$$\forall n \in \omega \;\; \mathrm{PA_F} \vdash \exists x > \overline{n}\, F(x).$$
- In particular:
$$\mathrm{PA_F} \vdash \exists x\, F(x). \tag{2}$$
- $(1) + (2) \Rightarrow \omega$-inconsistency.

The consistency of Yablo formulas

Theorem

PA_F *is consistent.*

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Proof.

- Take a nonstandard model $\mathcal{M}$ of PA.

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- Take a nonstandard model $\mathcal{M}$ of PA.
- Pick a nonstandard $a \in M$, let $A = \{a\}$.

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- Take a nonstandard model $\mathcal{M}$ of PA.
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- Put $F^{\mathcal{M}} = A$.

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- $\forall n \in \omega \ (\mathcal{M}, A) \models \neg F(n)$.

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- Take a nonstandard model $\mathcal{M}$ of PA.
- Pick a nonstandard $a \in M$, let $A = \{a\}$.
- Put $F^{\mathcal{M}} = A$.
- $\forall n \in \omega \; (\mathcal{M}, A) \models \neg F(n)$.
- But also, $(\mathcal{M}, A) \models \exists x \, F(x)$.

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- $\forall n \in \omega \ (\mathcal{M}, A) \models \neg F(n)$.
- But also, $(\mathcal{M}, A) \models \exists x \, F(x)$.
- Moreover, $\forall n \in \omega \ (\mathcal{M}, A) \models \exists x > n \, F(x)$.

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- But also, $(\mathcal{M}, A) \models \exists x\, F(x)$.
- Moreover, $\forall n \in \omega \ (\mathcal{M}, A) \models \exists x > n\, F(x)$.
- Hence $\forall n \in \omega \ (\mathcal{M}, A) \models F(n) \equiv \forall x > n\, \neg F(x)$ (both sides are false).

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- Put $F^{\mathcal{M}} = A$.
- $\forall n \in \omega \ (\mathcal{M}, A) \models \neg F(n)$.
- But also, $(\mathcal{M}, A) \models \exists x\, F(x)$.
- Moreover, $\forall n \in \omega \ (\mathcal{M}, A) \models \exists x > n\, F(x)$.
- Hence $\forall n \in \omega \ (\mathcal{M}, A) \models F(n) \equiv \forall x > n\, \neg F(x)$ (both sides are false).
- So $(\mathcal{M}, A) \models \mathrm{PA_F}$ and $\mathrm{PA_F}$ is consistent.

$\square$

Adding local disquotation

Definition

$AD = \{Tr(\overline{\ulcorner \varphi \urcorner}) \equiv \varphi \ : \varphi \in Sent_{\mathcal{L}}\}$

$YD = \{Tr(\overline{\ulcorner Y(\overline{n}) \urcorner}) \equiv Y(\overline{n}) \ : \ Y(\overline{n})$ belongs to the Yablo sequence$\}$.

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Definition ($\mathrm{PA_D}$)

PAT is obtained from PA by adding Tr (induction!)

$\mathrm{PA_D} = \mathrm{PAT} \cup AD \cup YD$.

$\mathrm{PA_D^-}$ is $\mathrm{PA_D}$ with induction without Tr.

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PAT is obtained from PA by adding *Tr* (induction!)
$\mathrm{PA_D} = \mathrm{PAT} \cup AD \cup YD$.
PA_D^- is $\mathrm{PA_D}$ with induction without *Tr*.

Theorem

$\mathrm{PA_D}$ *is* ω*-inconsistent.*

Adding local disquotation

Theorem

$\mathrm{PA_D}$ *is ω-inconsistent.*

Proof.

- Existence of YF entails:

$$\forall n \in \omega \;\; \mathrm{PA_D} \vdash Y(\overline{n}) \equiv \forall x > \overline{n} \, \neg Tr(\overline{\ulcorner Y(\dot{x}) \urcorner}).$$

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- By the inclusion of *YD* we get:

$$\forall n \in \omega \, \mathrm{PA_D} \vdash Tr(\overline{\ulcorner Y(\overline{n}) \urcorner}) \equiv \forall x > \overline{n} \, \neg Tr(\overline{\ulcorner Y(\dot{x}) \urcorner}).$$

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$$\forall n \in \omega \, \mathrm{PA_D} \vdash Tr(\overline{\ulcorner Y(\overline{n})\urcorner}) \equiv \forall x > \overline{n} \, \neg Tr(\overline{\ulcorner Y(\dot{x})\urcorner}).$$
- Let $F(x) := Tr(\overline{\ulcorner Y(\dot{x})\urcorner})$:
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- Let $F(x) := Tr(\overline{\ulcorner Y(\dot{x}) \urcorner})$:
$$\forall n \in \omega \ \ \mathrm{PA_D} \vdash F(\overline{n}) \equiv \forall x > \overline{n} \neg F(x).$$
- So $\mathrm{PA_D}$ contains $\mathrm{PA_F}$ (which is ω-inconsistent). □

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Proof.

- Take a nonstandard $\mathcal{M}$ of PA.
- Let $t(x) := \ulcorner Y(\dot{x}) \urcorner$. By overspill, there are nonstandard b and c such that $t^{\mathcal{M}}(b) = c$.

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Proof.

- Take a nonstandard $\mathcal{M}$ of PA.
- Let $t(x) := \ulcorner Y(\dot{x}) \urcorner$. By overspill, there are nonstandard b and c such that $t^{\mathcal{M}}(b) = c$.
- Let $Tr^{\mathcal{M}} = S = Th_{\mathcal{L}}(\mathcal{M}) \cup \{c\}$. Clearly, $(\mathcal{M}, S) \models AD$.

$$\forall n \in \omega \;\; (\mathcal{M}, S) \models \exists x > n \;\; Tr(\ulcorner Y(\dot{x}) \urcorner)$$

$$\forall n \in \omega \, (\mathcal{M}, S) \models \neg Y(\bar{n})$$

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- Let $Tr^{\mathcal{M}} = S = Th_{\mathcal{L}}(\mathcal{M}) \cup \{c\}$. Clearly, $(\mathcal{M}, S) \models AD$.
$$\forall n \in \omega \;\; (\mathcal{M}, S) \models \exists x > n \;\; Tr(\ulcorner Y(\dot{x}) \urcorner)$$
$$\forall n \in \omega \, (\mathcal{M}, S) \models \neg Y(\bar{n})$$
- Standard $Y(n)$ are not in S, so:
$$\forall n \in \omega \;\; (\mathcal{M}, S) \models \neg Tr(\ulcorner Y(\bar{n}) \urcorner).$$

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- Take a nonstandard $\mathcal{M}$ of PA.
- Let $t(x) := \ulcorner Y(\dot{x}) \urcorner$. By overspill, there are nonstandard b and c such that $t^{\mathcal{M}}(b) = c$.
- Let $Tr^{\mathcal{M}} = S = Th_{\mathcal{L}}(\mathcal{M}) \cup \{c\}$. Clearly, $(\mathcal{M}, S) \models AD$.

$$\forall n \in \omega \ \ (\mathcal{M}, S) \models \exists x > n \ \ Tr(\ulcorner Y(\dot{x}) \urcorner)$$

$$\forall n \in \omega \, (\mathcal{M}, S) \models \neg Y(\bar{n})$$

- Standard $Y(n)$ are not in S, so:

$$\forall n \in \omega \ \ (\mathcal{M}, S) \models \neg Tr(\ulcorner Y(\bar{n}) \urcorner).$$

- So $(\mathcal{M}, S) \models YD$ (UYD fails here). $\square$

The consistency of $\mathtt{PA_D}$

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The consistency of $\mathsf{PA_D}$

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Proof.

By finite satisfiability (put only the last Yablo sentence in the extension of *Tr*, check induction holds), and compactness. □

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- For a nonstandard $\mathcal{M}$ of PA, $\mathcal{M} \models \neg\varphi$.

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Proof.

- Suppose $\mathsf{PA} \nvdash \varphi$.
- So $\mathsf{PA} \cup \{\neg\varphi\}$ is consistent.
- For a nonstandard $\mathcal{M}$ of PA, $\mathcal{M} \models \neg\varphi$.
- There is an elementarily equivalent $\mathcal{M}'$ such that $(\mathcal{M}', Tr^{\mathcal{M}'}) \models \mathsf{PA}_{\mathsf{D}}$.

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- So $\mathsf{PA} \cup \{\neg\varphi\}$ is consistent.
- For a nonstandard $\mathcal{M}$ of PA, $\mathcal{M} \models \neg\varphi$.
- There is an elementarily equivalent $\mathcal{M}'$ such that $(\mathcal{M}', Tr^{\mathcal{M}'}) \models \mathsf{PA_D}$.
- $(\mathcal{M}', Tr^{\mathcal{M}'}) \not\models \varphi$, and so $\mathsf{PA_D} \nvdash \varphi$. $\square$

Uniform Yablo Disquotation yields contradiction

Definition

$$UYD = \forall x(Tr(\overline{\ulcorner Y(\dot{x})\urcorner}) \equiv Y(x))$$

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- So $\forall x\,(Y(x) \equiv \exists w > x\; Y(w))$
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Local disquotation with ω-rule is inconsistent

Theorem

Let $\mathtt{PA}_D^{\omega-} = (\mathtt{PAT}^- \cup \mathtt{AD} \cup \mathtt{YD})^\omega$. $\mathtt{PA}_D^{\omega-}$ *is inconsistent.*
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- $\mathsf{PA}_D^{\omega-} \vdash \forall x \neg Tr(\overline{\ulcorner Y(\dot{x}) \urcorner})$ [ω-rule]
- In particular: $\mathsf{PA}_D^{\omega-} \vdash Y(\overline{23})$

$\square$

Classical set-up vs. Yablo

- Even those theories which prove the existence of Yablo sentences are still consistent.
- They're ω-inconsistent with local Yablo disquotation, though.
- One way to obtain a contradiction: uniform Yablo disquotation.
- Another one: local disquotation and $\omega - rule$.

sl-semantics (Mostowski, 2001a,b, 2016)

Definition (*FM*-domains)

Take a relational arithmetical language.

$$FM(\mathbb{N}) = \{\mathbb{N}_n : \ n = 1, 2, \ldots\}$$

$$\mathbb{N}_n = (\{0, 1, \ldots, n-1\}, +^{(n)}, \times^{(n)}, 0^{(n)}, s^{(n)}, <^{(n)}).$$

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Definition ($FM(\mathbb{N})^T$)

An $FM(\mathbb{N})^T$-domain is a set of $(\mathbb{N}_k, T_k)$ containing a unique member for each $k \in \omega$, where $T_k \subseteq \{0, \ldots, k-1\}$.

Things that are kinda the same

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- Syntax is still representable.

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Theorem (sl-Yablo existence)

There exists a formula $Y(x)$ s.t. for any $FM(\mathbb{N})^T$-domain:

$$\forall n \in \omega \;\; FM(\mathbb{N})^T \models_{sl} Y(n) \equiv \forall x\,(x > n \Rightarrow \neg Tr(\ulcorner Y(\dot{x})\urcorner))$$

YS are non-trivially false in the limit

Theorem

For any class $\mathcal{K}$ of finite models, if $\mathcal{K} \models_{sl} AD + YD$, then:
$\forall\, n \in \omega\ \mathcal{K} \models_{sl} \neg Y(n)$. $\mathtt{AD}$ *is not essential.*

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The standard argument still flies, *mutatis mutandis.* $\square$

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Reason.

In each point take truth to refer to all existing codes of true arithmetical formula, and the code of the last YS. □

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Fact (Cheap shot)

- *n is the greatest number sl-fails, for any n.*

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Modal interpretation of quantifiers (Urbaniak, 2016)

Definition (Accessibility relation in FM-domains)

$R(M,N)$ iff $M \subseteq N$.
For $\mathbb{N}_m, \mathbb{N}_n \in FM(\mathbb{N})$ this boils down to $m \leq n$.

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Example

$$(\exists x \, \forall y \; x \geq y) \in sl(FM(\mathbb{N})), \notin msl(FM(\mathbb{N}))$$

Arithmetic regained

Theorem

$$msl(FM(\mathbb{N})) = Th(\mathbb{N})$$

Reason.

- Finite points are submodels of $\mathbb{N}$.
- Q-free φ are preserved for parameters in a point.
- $\exists x\, \varphi$ true in $\mathbb{N}$ has a finite witness, which belongs to some finite point. $\square$

YS & the modal interpretation

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If $YD \subseteq msl(FM(\mathbb{N})^Y)$, *then:*

$$\forall n \in \omega \;\; Y(n) \notin msl(FM(\mathbb{N})^Y)$$

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- $\exists q \geq p \exists b \in (a,q) \, \mathbb{N}_q \models_m Tr(Y(b))$ Contradiction. $\square$

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No LYD

Theorem

There is no $FM(\mathbb{N})^Y$-domain such that $YD \subseteq msl(FM(\mathbb{N})^Y)$.

Proof.

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- Pick witness $a > 0$. $Y(a) \in msl(FM(\mathbb{N})^Y)$. Contradiction! $\square$

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m-semantics

Arithmetic regained, adding LAD and LYD gives inconsistency.
UYD or ω-rule are not needed.

Thank you! Questions?

Finite model *in concreto*

A finite sequence of finite books each saying that all the ones behind it are false. The last one is right.
(Or so we like to think.)

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