Ordinal analysis of Kripke-Platek set theory via Schmerl formula

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Schmerl formula

The arithmetical Π^0_n uniform reflection for theory T

$$\mathsf{RFN}_{\Pi^0_n}(T) : (\forall \varphi \in \Pi^0_n)(\mathsf{Prv}_T(\varphi) \to \mathsf{Tr}_{\Pi^0_n}(\varphi)).$$

For recursive ordinals α we define r.e. theories $\mathsf{RFN}^{\alpha}_{\Pi^0_n}(T)$:

$$\mathsf{RFN}^{\alpha}_{\Pi^0_n}(T) = T + \{\mathsf{RFN}_{\Pi^0_n}(\mathsf{RFN}^{\beta}_{\Pi^0_n}(T)) \mid \beta < \alpha\}.$$

Formally definition is carried out using Fixed Point Lemma.

EA is a weak fragment of PA proving totality of exponentiation.

Schmerl formula:

$$\mathsf{RFN}^{\alpha}_{\Pi^0_{n+1}}(\mathsf{EA}) \equiv_{\Pi^0_n} \mathsf{RFN}^{\omega^\alpha}_{\Pi^0_n}(\mathsf{EA}), \text{ for } \alpha > 0.$$

Classifying Π^0_2 consequences of PA in terms of iterated reflection

Ordinal $\omega_n = \underbrace{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}_{n \text{ times}}$.

$$\mathsf{PA} \equiv \bigcup_{n\in\mathbb{N}} \mathsf{RFN}_{\Pi^0_n}(\mathsf{EA}).$$

$$\Downarrow$$

$$\mathsf{PA} \equiv_{\Pi^0_2} \bigcup_{n\in\mathbb{N}} \mathsf{RFN}^{\omega_n}_{\Pi^0_2}(\mathsf{EA}).$$

$$\Downarrow$$

$$\mathsf{PA} \equiv_{\Pi^0_2} \mathsf{RFN}^{\varepsilon_0}_{\Pi^0_2}(\mathsf{EA}).$$

From reflection to fast-growing functions

$f_\alpha(x)$ is α'th function from fast-growing hierarchy

For any Δ_0^0 formula $\varphi(x, y)$:

$$\mathrm{RFN}^{\alpha}_{\Pi_2^0}(\mathrm{EA}) \vdash \forall x \exists y \; \varphi(x, y)$$

$$\Downarrow$$

$$\mathrm{RFN}^{\alpha}_{\Pi_2^0}(\mathrm{EA}) \vdash \forall x (\exists y < f^{n}_{2+\beta}(x)) \varphi(x, y), \text{ for some } \beta < \alpha \text{ and } n \in \mathbb{N}$$

Hence

$$\mathrm{PA} \vdash \forall x \exists y \; \varphi(x, y)$$

$$\Downarrow$$

$$\mathrm{PA} \vdash \forall x (\exists y < f_{\alpha}(x)) \varphi(x, y), \text{ for some } \alpha < \varepsilon_0$$

KPω vs PA

Axioms of KP are: Extensionality, Pair, Union, Δ_0-Separation, Δ_0-Collection, and Foundation.
KPω is KP + Infinity.

Transitive models of KP are known as admissible sets.
Analogies between PA and KPω:

PA	KPω
$\mathbb{N}$	admissible sets with ω
r.e. sets	Σ_1 classes
recursive functions	Σ_1 functions
recursive ordinal notations	Δ_0 class well-orderings
ω	$\mathbf{On}$
ε_0	$\varepsilon_{\mathbf{On}+1}$
hierarchies of recursive functions	hierarchies of Σ_1 functions $\mathbf{On} \to \mathbf{On}$
r.e. theories	class-theories with Σ_1 class of axioms

Reflection principles in KP

The axioms of KP_0 are: Extensionality, Pair, Union, Δ_0-Separation, Δ_0-Collection, Regularity, Transitive Containment, and Totality of Rank Function.

Definitions inside KP_0:
$\mathbf{L}$ is usual set-theoretic language with constants c_s, for all sets s
Note: $\mathbf{L}$ *forms a proper class*
$\mathbf{\Pi}_n, \mathbf{\Sigma}_n, \mathbf{\Delta}_0$ are $\Pi_n, \Sigma_n, \Delta_0$ with set constants.

Let T be $\mathbf{L}$ theory given by $\mathbf{\Sigma}_1$ formula defining its class of axioms.

$$\mathrm{RFN}_{\mathbf{\Pi}_n}(\mathrm{KP}_0\omega) : (\forall \varphi \in \mathbf{\Pi}_n)(\mathrm{Prv}_T(\varphi) \to \mathrm{Tr}_{\mathbf{\Pi}_n}(\varphi)).$$

$\mathbf{a}$ ranges over $\mathbf{\Delta}_0$ class well-orderings.
Schmerl formula:

$$\mathrm{RFN}^{\mathbf{a}}_{\mathbf{\Pi}_{n+1}}(\mathrm{KP}_0\omega) \equiv_{\mathbf{\Pi}_n} \mathrm{RFN}^{\omega^{\mathbf{a}}}_{\mathbf{\Pi}_n}(\mathrm{KP}_0\omega), \text{ for } \mathbf{a} > 0.$$

Reformulating KPω in terms of iterated reflection

The class well-ordering $\omega_n^{\mathbf{a}} = \underbrace{\omega^{\cdot^{\cdot^{\cdot^{\omega^{\mathbf{a}}}}}}}_{n \text{ times}}$

Over $\mathsf{KP}_0\omega$ Foundation is equivalent to $\mathbf{On}+1$-iterated reflection:

$$\mathsf{KP}\omega \equiv \bigcup_{n\in\mathbb{N}} \mathsf{RFN}^{\mathbf{On}+1}_{\mathbf{\Pi}_n}(\mathsf{KP}_0\omega).$$

$$\Downarrow$$

$$\mathsf{KP}\omega \equiv_{\Pi_2} \bigcup_{n\in\mathbb{N}} \mathsf{RFN}^{\omega_n^{\mathbf{On}+1}}_{\mathbf{\Pi}_2}(\mathsf{KP}_0\omega).$$

$$\Downarrow$$

$$\mathsf{KP}\omega \equiv \mathsf{RFN}^{\varepsilon_{\mathbf{On}+1}}_{\mathbf{\Pi}^0_2}(\mathsf{KP}_0\omega).$$

Hierarchies of ordinal function

We have assignment of fundamental sequences $\mathbf{a}[\xi]$, for $\mathbf{a} < \varepsilon_{\mathbf{On}+1}$.

$$\mathbf{a} = \sup_{\xi < \tau_{\mathbf{a}}} \mathbf{a}[\xi], \text{ where } \tau_{\mathbf{a}} \leq \mathbf{On}$$

Bachmann defined extension of Veblen hierarchy $\varphi_{\mathbf{a}}$.
We use similar hierarchy $\mathbf{F}_{\mathbf{a}}$ that is closely connected to fast-growing hierarchy:

$f_\alpha \colon \mathbb{N} \to \mathbb{N}$	$\mathbf{F}_{\mathbf{a}} \colon \mathbf{On} \to \mathbf{On}$
$f_0(n) = n + 1$	$\mathbf{F}_0(\alpha) = \alpha + 1$
$f_{\alpha+1}(n) = f_\alpha^n(n)$	$\mathbf{F}_{\mathbf{a}+1}(\alpha) = \sup_{n<\omega} \mathbf{F}_{\mathbf{a}}^n(\alpha)$
	$\mathbf{F}_{\mathbf{a}}(\alpha) = \sup_{\xi < \tau_{\mathbf{a}}} \mathbf{F}_{\mathbf{a}[\xi]}(\alpha)$ if $\tau_{\mathbf{a}} < \mathbf{On}$
$f_\lambda(n) = f_{\lambda[n]}(n)$	$\mathbf{F}_{\mathbf{a}}(\alpha) = \mathbf{F}_{\mathbf{a}[\alpha]}(\alpha)$ if $\tau_{\mathbf{a}} = \mathbf{On}$

Ordinal bounds for Π_2 theorems of KPω

Recall that

$$\mathsf{KP}\omega \equiv_{\mathbf{\Pi}_2} \mathsf{RFN}^{\varepsilon_{\mathbf{On}+1}}_{\mathbf{\Pi}_2}(\mathsf{KP}_0\omega).$$

$$\mathsf{RFN}^{\mathbf{a}}_{\mathbf{\Pi}_2}(\mathsf{KP}_0\omega) \vdash \text{“}\mathbf{F_b} \text{ is total”, for } \mathbf{b} < 1 + \mathbf{a}$$

For any $\mathbf{\Delta}_0$ formula $\varphi(x, y)$

$$\mathsf{RFN}^{\mathbf{a}}_{\mathbf{\Pi}_2}(\mathsf{KP}_0\omega) \vdash \forall x \exists y\, \varphi(x, y)$$
$$\Downarrow$$
$$\mathsf{RFN}^{\mathbf{a}}_{\mathbf{\Pi}_2}(\mathsf{KP}_0\omega) \vdash \forall x \exists y \big(\mathrm{rk}(y) \leq \mathbf{F}^n_{1+\mathbf{b}}(\mathrm{rk}(x)) \wedge \varphi(x, y)\big)$$
and
$$\mathsf{RFN}^{\mathbf{a}}_{\mathbf{\Pi}_2}(\mathsf{KP}_0\omega) \vdash \forall x \exists y \big(y \in L_{\mathbf{F}^n_{1+\mathbf{b}}(\mathrm{rk}(x))}(x) \wedge \varphi(x, y)\big)$$
for some $\mathbf{b} < \mathbf{a}$ and $n \in \mathbb{N}$

Thank You!