Takeuti's finitism and his proof of the consistency of arithmetic

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Finitist proofs of consistency

Let's outline Gentzen's proof of the consistency of PA.

He designs a system of ordinals and an ordering of these ordinals that are each concrete and thus finitistically acceptable.

This ordering has type ϵ_0.

Proofs in PA are assigned these ordinals according to the rules of inference used.

Gentzen gives a procedure for reducing proofs so that each proof of inconsistency gets reduced to another proof of inconsistency with a smaller ordinal.

If there is a proof of inconsistency, this procedure generates an infinitely decreasing sequence of such ordinals.

By the well-ordering of the ordering of type ϵ_0, such a sequence is impossible.

Thus there is no proof of inconsistency in PA.

Tarski, “Contribution to the discussion of P. Bernays *Zur Beurteilung der Situation in der beweistheoretischen Forschung*” (1954)

Gentzen’s proof of the consistency of arithmetic is undoubtedly a very interesting metamathematical result, which may prove very stimulating and fruitful. I cannot say, however, that the consistency of arithmetic is now much more evident to me (at any rate, perhaps, to use the terminology of the differential calculus more evident than by an epsilon) than it was before the proof was given.

Girard, *The Blind Spot* (2011)

Concerning Gentzen's second consistency proof, André Weil said that "Gentzen proved the consistency of arithmetic, i.e., induction up to the ordinal ω, by means of induction up to ϵ_0", the venom being that ϵ_0 is much larger than ω.

Hilbert & Bernays, *Grundlagen der Mathematik*, Vol. 1 (1934)

Our treatment of the basics of number theory and algebra was meant to demonstrate how to apply and implement direct contentual inference that takes place in thought experiments [*Gedanken-Experimenten*] on intuitively conceived objects and is free of axiomatic assumptions. Let us call this kind of inference “finitist” inference for short, and likewise the methodological attitude underlying this kind of inference as the “finitist” attitude or the “finitist” standpoint.. . . With each use of the word “finitist”, we convey the idea that the relevant consideration, assertion, or definition is confined to objects that are conceivable in principle, and processes that can be effectively executed in principle, and thus it remains within the scope of a concrete treatment.

Hilbert, “Die Grundlagen Der Elementaren Zahlentheorie” (1931)

This is the fundamental mode of thought that I hold to be necessary for mathematics and for all scientific thought, understanding, and communication, and without which mental activity is not possible at all.

Tait, “Finitism” (1981)

[Finitistically acceptable reasoning] is a minimal kind of reasoning presupposed by all non-trivial mathematical reasoning about numbers.

Hilbert & Bernays, *Grundlagen der Mathematik*, Vol. 1 (1934)

Regarding this goal [of proving consistency], I would like to emphasize that an opinion, which had emerged intermittently—namely that some more recent results of Gödel would imply the infeasibility of my proof theory—has turned out to be erroneous. Indeed, that result shows only that—for more advanced consistency proofs—the finitist standpoint has to be exploited in a manner that is sharper [*schärferen*] than the one required for the treatment of the elementary formulations.

In Gentzen's proof every step except the well-ordering of the ordering of type ϵ_0 can be effected in primitive recursive arithmetic (generally accepted to be finitistically acceptable).

In particular, it needs to be proved that every strictly decreasing computable sequence of ordinals in this ordering is finite.

This is the part of the proof that needs to be justified from the finitist standpoint.

Hilbert & Bernays, *Grundlagen der Mathematik*, Vol. 2 (1939)

The question arises as to whether finitary methods are in a position to exceed the domain of inferences formalizable in Z_μ. This question is admittedly, as so formulated, not precise; because we have introduced the expression "finitary" not as a sharply delimited endpoint, but rather as a designation of a methodological guideline, which would enable us to recognize certain kinds of concept formation and certain kinds of inferences as definitely finitary and others as definitely not finitary, but which however delivers no exact separating line between those which satisfy the demands of the finitary method and those which do not.

Takeuti's argument

Takeuti, “Consistency Proofs and Ordinals” (1975)

Gödel’s incompleteness theorem has changed the meaning of Hilbert’s program completely. Because of Gödel’s result consistency proofs now require a method that is finite (or constructive) but which is nevertheless very strong when formalized. People think this is impossible or at least unlikely and extremely difficult. The situation is somewhat similar to that of finding a new axiom that carries conviction and decides the continuum hypothesis.

The claim about decreasing sequences of ordinals has the provability strength of the consistency of PA, but is still, Takeuti alleges, finitistically acceptable.

Takeuti calls an ordinal μ accessible if it has been finitistically proved that every strictly decreasing sequence starting with μ is finite.

This is the step in Gentzen's proof that needs to be finitistically justified: that every ordinal up to ϵ_0 is accessible.

Takeuti observes that it is clear that every natural number is accessible.

The crux of his argument is to extend this observation to infinite ordinals.

Firstly, he argues that $\omega + \omega$ is accessible: the first term μ_0 of any decreasing sequence from $\omega + \omega$ is either of the form n or $\omega + n$.

If the former, then we're done.

If the latter, then consider the sequence
$\mu_{n+1} < \cdots < \mu_2 < \mu_1 < \mu_0$.

This sequence has length $n + 2$ and thus μ_{n+1} must be a natural number.

Such reasoning will also show that ordinals to ω^ω are accessible.

For ordinals written in Cantor normal form up to ϵ_0, Takeuti explains how to continue this reasoning.

It is crucial that each of these steps can be shown by a finitistically acceptable argument.

That is, they must be "effectively executed in principle…within the scope of a concrete treatment".

We are meant to see this by Gedankenexperimenten.

But are these steps really thinkable in an effective, concrete way?

This must be confirmed in order to accept Takeuti's proof of the consistency of arithmetic as finitary.

Challenges to Takeuti

The system of ordinal notations used here, Kleene's O, is known to be a Π^1_1 set.

Can we finitistically prove things about terminating decreasing sequences in O in light of this set's complexity?

Rathjen (2014) has noted that the accessibility of ϵ_0 used in Takeuti's proof of consistency is only Π^0_2.

Takeuti, “Axioms of Arithmetic and Consistency” (Sugaku Seminar, 1994)

There is not much reason to oppose this idea by claiming that the notion that all decreasing sequences terminate within finite steps is a Π^1_1 notion in Kleene’s hierarchy. What is important is not which hierarchy the notion belongs to, but how clear it is.

Takeuti, “Consistency Proofs and Ordinals” (1975)

This proof is very clear and transparent if one is familiar with the primitive recursive structure of the ordinals less than ϵ_0.

Gentzen, “Die Widerspruchsfreiheit der reinen Zahlentheorie” (1936)

We might, for example, visualize the initial cases with the characteristics $1, 2, 3$ in detail. As the characteristic grows, nothing new is *basically* added; the method of progression always remains the same. It must of course be admitted that the complexity of the multiply-nested infinities which must be ‘run through’ grows considerably; this running through must always be regarded as ‘potential’…The difficulty lies in the fact that although the *precise finitist sense* of the ‘running through’ of the ρ-numbers is reasonably perspicuous in the initial cases, it becomes of such great *complexity* in the general case that it is only remotely visualizable…

Gödel, "On an extension of finitary mathematics which has not yet been used" (1972)

The situation may be roughly described as follows: Recursion for ϵ_0 could be proved finitarily if the consistency of number theory could. On the other hand the validity of this recursion can certainly not be made *immediately* evident, as is possible, for example in the case of ω^2. That is to say, one cannot grasp at one glance the various structural possibilities which exist for decreasing sequences, and there exists, therefore, no *immediate* concrete knowledge of the termination of every such sequence. But furthermore such concrete knowledge (in Hilbert's sense) cannot be realized either by a stepwise transition from smaller to larger ordinal numbers, because the concretely evident steps, such as $\alpha \to \alpha^2$, are so small that they would have to be repeated ϵ_0 times in order to reach ϵ_0.

Gödel, "Über eine bisher noch nicht benützte Erweiterung des finite Standpunktes" (1958)

It cannot be determined out of hand whether the need for abstract notions is due merely to the practical impossibility of our intuitively imagining states of affairs that are all too complex from the combinatorial point of view or whether there are theoretical reasons for it.

This “practical” problem means that some knowers may be able to have “immediate concrete knowledge” of the termination of some sequences, but not of others ; while other knowers may be yet more capable of such “intuitions”.

One person might intuit accessibility up to, say, ω^2; another to ω^ω; another to ϵ_0.

If this is right, then Takeuti’s sharpening of finitism loses the quality stressed by Hilbert, that finitary reasoning is the core type of reasoning common to all scientific knowledge, and hence knowers.

But here we recognize, as did Gödel, that the problem arises from identifying intuition and visualizability.

Letter of Gödel to Bernays, 25 July 1969

Hilbert's finitism (through the requirement of being "intuitive" [Anschaulichkeit]) has a quite unnatural boundary.

This identification is a remnant of the classical Kantian understanding of intuition as (something like) visualization, which is to be contrasted with the abstract apprehension characteristic of logical knowledge.

New horizons open when we recognize that Takeuti came from a considerably different background from Hilbert, Bernays, and Gödel.

He was a Japanese thinker with relatively little access to Western texts until he came to the USA.

In fact, questions about knowledge from finite and infinite standpoints, of the concrete and the abstract, of the intuitive and the logical, were already being studied in Japan in the 1910s, before Hilbert's seminal work on these questions that gave birth to proof theory.

There is good reason to think that Takeuti was knowledgable about this work.

Takeuti as a Japanese philosopher

Takeuti, “Proof theory and set theory” (1985)

Foundational problems begin when we realize that we cannot examine infinitely many objects one by one. However, it is very easy for us to imagine an infinite mind which can do so. Actually by working in mathematics we have been building up our intuition on what an infinite mind can do. An infinite mind must be able to operate on infinitely many objects as freely as we operate on finitely many objects. Thus it can unite members of a set D to form arbitrary subsets of D. It can examine each of these subsets and so on.

Takeuti, “About mathematics” (Japanese, 1972)

Speaking more clearly, modern mathematics or modern set theory is our hypothesis or conjecture about infinite mind.

Takeuti explains that testing this “conjecture” leads to three problems for the foundations of mathematics.

1. To formulate the function of infinite mind.
2. To justify our intuition of the world of infinite mind using only our finite mind.
3. To formulate the function of finite mind.

Takeuti, “Proof theory and set theory” (1985)

A long time ago when I discussed my standpoint with Gödel, I used different terminology. It was Gödel who suggested the phrase “infinite mind”, and “infinite mind” became standard terminology in our discussions.

While Takeuti does not say what terminology he previously used, such notions had been studied in Japan in the early twentieth century.

Takeuti mentions some familiarity with the works of Suetsuna, a number theorist who wrote on philosophy of mathematics and later on Kegon Buddhism; Suetsuna pursued a kind of finitism.

Suetsuna was heavily influenced by Nishida, the foremost Japanese philosopher of the twentieth century.

In work starting in the 1910s, Nishida contrasted a logical standpoint from a mathematical standpoint, where the latter involves concrete intuitions of wholes, rather than the parts that arise from logical analysis.

He held that we cannot intuit the infinite as a whole, since we are finite.

But we can self-reflect: we can, in intuition, make ourselves the object of our thought, and then make the content of that intuition into a further object, in intuition; and so on.

Following Royce, Nishida elaborated on this Dedekindian construction, talking of drawing a map of where you are now, point by point, including the map you are drawing.

Nishida called this capacity “self-awareness” [jikaku 自覚] and maintained that the resulting object is the “true sense of infinity” (“Understanding in logic and in mathematics”, 1912).

Takeuti, “Proof theory and set theory” (1985)

We believe that the set universe is a growing universe i.e., the creation of sets by the infinite mind is always in process and never finished. The picture that the infinite mind is endlessly creating sets starting with the empty set provides us with some reason to justify the axioms of set theory. Moreover the infinite mind, as a mind, reflects on what he is doing. So he can imagine the stage when he finishes his creation and starts it again after that.

Note the following passage of the *Avatamsaka-sutra*, a key text in the Kegon Buddhism of Nishida:

The Buddha is also like the mind,
and living beings are like the Buddha.
It must be known that the Buddha and the mind are, in their essence, inexhaustible.

If one understands that the activity of the mind creates the worlds everywhere,
he will see the Buddha,
and understand the real nature of the Buddha.

Takeuti, “Proof theory and set theory” (1985)

Gentzen’s proof is an assurance from the finite standpoint of the truth as conceived by the infinite mind.

The philosophical upshot of this reading is that we should try to understand Takeuti’s finitism in the light of Japanese thought as much as the usual writing on finitism that draws on Kant as understood by Hilbert & Bernays, for example.

A mathematical upshot of this reading is that Takeuti's finitism, more than Hilbert & Bernays, can provide for a hierarchy of finitisms.

We can stratify these new finitisms by quantifier complexity.

But not just any quantification, but only on ordinals, which have a nice structure (the "true" structure of infinity) and which do not require checking through a disorderly set like the set of all informal infinitary proofs.

At least this is an idea.